PARAMETRIC VIBRATIONS OF AN ENGINE - VALVE SYSTEM

by

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CERTIFICATE

This is to certify that the work entitled ''Parametric Vibrations of an Engine-Valve System'' by Santiranjan Pramanik has been carried out under my supervision and has not been submitted elsewhere for a degree.

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NOTATIONS AND SYMBOLS

a	-	non-dimensional amplitude of response
a_o, a_n, b_n	-	Fourier coefficients in F.S. representation of $\overline{B}_1(t)$
a'_n, b'_n	-	Fourier coefficients in F.S. representation of $\overline{B}_2(t)$
$A, \overline{A}$	-	Complex Functions
$B(t)$	-	amplitude of first mode for one mode approximation
$B_1(t), B_2(t)$	-	amplitudes of first and second mode respectively for two mode approximation
$\overline{B}(t), \overline{B}_1(t)$ $\overline{B}_2(t)$	-	non dimensional quantities
c	-	viscous damping coefficient
E	-	Youngs Modulus of the beam
D_o, D_1	-	Partial derivatives with respect T_o and T_1 respectively.
$F(t)$	-	Sum of the spring force and inertia force exerted on the concentrated mass
F_o	-	preload on the spring
$g(t)$	-	displacement excitation
$\overline{g}(t)$	-	non-dimentional displacement excitation
I	-	moment of inertia of the beam
K_f	-	spring stiffness

K_{f*}	-	ratio of spring stiffness to the transverse tip stiffness of the beam
L	-	Total length of the undeformed beam
L_p	-	projected length of the elastic curve of the beam on the vertical axis
M_b, M_1, M_2, M_3	-	Bonding moments
M	-	Concentrated mass
M_*	-	ratio of concentrated mass to beam mass
P_{cr}	-	critical load of the beam
q_b, q	-	frequency ratios for backbone curve and response curve respectively
R	-	horizontal reaction force at the top end of the beam
s	-	arc coordinate along the elastic curve
T_o, T_1	-	time scale
t	-	time
v	-	lateral displacement of the beam at any cross-section
z	-	time

Greek Symbols:

β, η	-	arc coordinates
μ	-	beam mass per unit length
ω	-	circular frequency of excitation
ω_o	-	fundamental eigenfrequency of the beam
ω_{ol}	-	fundamental eigen frequency of the beam with axial compression F_o

G	–	nondimensional amplitude of displacement excitation
σ	–	detuning parameter

ABSTRACT

The objective of this thesis is to find out the parametric instability regions and steady state response characteristics in the neighbourhood of main parametric resonance of a vertical beam which has a concentrated mass and is restrained at the upper end and simultaneously subjected to a harmonic axial displacement excitation at the bottom end. This system is a dynamic model of an engine-valve mechanism where the displacement excitation is produced by a cam. The beam under this loading is considered to be executing lateral vibration with large amplitude which results in longitudinal displacement of any section and the axial displacement of the end load. The equation of motion for this lateral vibration of the beam is deduced. This has non-linear terms because of large curvature and longitudinal inertia of the beam elements and the concentrated mass. Two mode approximation for the lateral vibration of the beam is made and Galerkin's method is applied to reduce the governing partial differential equation to two coupled non-linear ordinary differential equations with periodic coefficients. These equations are linearised and then the instability regions are found out by the use of Hills determinants. The principal parametric instability region of these equations is compared with the corresponding region obtained from

the linearised equation of one mode approximation.

For the steady state response characteristics of one mode approximation in the neighbourhood of its principal parametric resonance the methods of (1) harmonic balance (2) slowly varying parameters and (3) multiple scales have been used and these results are mutually compared. The effects of various parameters like the ratio of longitudinal spring stiffness to the transverse tip stiffness of the beam, ratio of the concentrated mass to the beam mass and the non-dimensional amplitude of the displacement excitation (with respect to beam length) on the response curve are investigated.

CHAPTER 1

INTRODUCTION

In contrast with the forced oscillation of a single or multi-degree of freedom system (mechanical or electrical) in which excitation appears as inhomogeneities in the governing differential equation, in the parametically excited systems the excitation appears as the coefficients in the governing differential equation. So for these systems one is led to differential equations with time varying coefficients. Since the excitations when they are time independent appear as parameters in the governing differential equation, this type of excitation is called parameteric excitation. Moreover, in constrast with the external excitation in which a small excitation cannot produce large response unless the frequency of excitation is close to one of the natural frequencies of the system, a small parameteric excitation can produce a large response when the frequencies of the excitation are other than the natural frequencies of the system. Moreover, parameteric instability occurs over a region of a parameter space consisting of excitation amplitude and frequency and not at discrete points. This instability can correspond to large displacement in the direction normal to the longitudinal excitation of finite amplitude.

Parametrically excited systems, reducible to the standard Mathieu Equation having two parameters (δ and ϵ), have some boundary curves in the parameter plane (δ,ϵ) which separate the stable and unstable regions. The stable and unstable regions are of interest because one wants to avoid the unstable regions so that there is no possibility of very large response of the system.

There are a number of analytic techniques available for the determination of the boundaries separating stability from instabilities. These techniques can be divided broadly into following three classes-

(i) Hills Method of Infinite Determinant

(ii) Perturbation Methods

(iii) Liapunov's Theory.

The Hill's method of infinite determinants is extensively used for a single degree of freedom system and has been recently applied to multidegree of freedom systems by a number of researchers [1,2]. The second class consists of perturbation methods [16] which assume that the variable coefficient terms are small in some sense. It is also necessary to assume a series solution of the differential equation in powers of a small dimensionless parameter ϵ and make sure that it is a uniformly valid exponsion which means that the higher order terms in ϵ should provide a small correction to the lower order terms of ϵ .

The third class uses Liapunov's theory [15]. This approach is limited by the ability to find out a suitable Liapunov function.

The influence of damping on the instability boundaries was discussed by many researchers [3,4,5]. In most cases, damping force has stabilizing effect but Valeev [6] showed that for certain combination resonances the damping force may alter a stable state into an unstable one. Moreover, Stevens [8] showed that some viscoelastic materials are destabilizing.

By deleting all the non-linear terms in the governing equation of the system, one arrives at a differential equation, called the corresponding linear system. It plays a key role in the analysis of weakly non-linear systems. One can write the solution of a non-linear system in the form of a perturbation series which may be considered as a development in the neighbourhood of the solution of the corresponding linearised equation. There are a number of ways in which this perturbation can be effected. One starts with straightforward expansion, which is not uniformaly valid. So several modifications of the straight forward procedure are to be done so as to lead to uniformly valid expansion. One modification of it is the Lindsted-Poincare Method where a new time scale is introduced to permit the frequency and amplitude to interact and there are provisions to avoid secular terms (which give, unbounded solution). But this

method can not treat damped systems conveniently. So method of multiple scales is used as a further modification of Lindsted -Poincare's Method. The underlying idea of the method of multiple scales is to consider the expansion respresenting the response to a function of multiple independent variables or scales, instead of a single variable. The steady state vibration can also be analysed by the method of hormonic balance and by method of slowly varying parameters.

BRIEF REVIEW OF AVAILABLE LITERATURE:

Parametric instability was first observed by Michael Faraday in 1831 [7], when he found that surface waves in a fluid-filled cylinder under vertical excitation exhibited twice the period of excitation itself. Melde [7] in 1859 found that though a longitudinal force is applied on a string tied between a rigid support and a heavy tuning fork, it exhibits a lateral response. Beliaev in 1924 first considered the parameteric instability of a column pinned at both ends and subjected to a time dependent axial force [7;19]. He reduced the governing equation of motion to a standard Mathieu-Hill equation and showed the stability and instability regions in the parameter space. Later many other scientists extended his work, Bolotin [3] considered various kinds of structural elements and discussed their dynamic stability. He derived a modified equation of motion for a column considering the initial curvature and longitudinal inertia of the column elements and the attached end mass.

Evensen and Evan- Iwanowski [9] and Handoo and Sundararajan [10] investigated the parametric instability regions of a perfect column with an attached end mass both analytically and experimentally. They considered the longitudinal inertia of the beam and the attached mass. Carlson [11] considered a tensioned bar with initial curvature and an elastic spring restraint at an end which included a lumped mass. He found the steady state response characteristics by the method of harmonic balance. Sato and Saito [12] investigated the steady state response of parametric vibration of a simply supported horizontal beam, carrying a concentrated mass which is not at the end under the influence of gravity and periodic longitudinal exciting force. They included non-linear terms arising from moderately large curvatures, longitudinal inertia of the beam and concentrated mass and rotatory inertia of the concentrated mass in the equation of motion. They applied harmonic balance method to investigate the dynamic response. Saito and Koizumi [13] investigated the steady state response of parametric vibration of a simply supported horizontal beam with a concentrated mass at one end and subjected to a peroidic axial displacement excitation at the other end under the influence of gravity. They considered the non-linear terms arising from longitudinal inertia of concentrated end mass and beam element. They also used harmonic balance method for finding the steady state behaviour. Gürgöze [14] found out the principal parametric instability

region and steady state response of a simply supported vertical beam by Mettler's method and method of harmonic balance respectively. The beam carries a concentrated end mass and is restrained at one end and subjected to a periodic axial displacement excitation at the other end. He considered one mode approximation of its lateral vibration and included non-linear terms arising from moderately large curvature and longitudinal inertia of the concentrated mass and beam element.

SCOPE OF THE PRESENT WORK:

In the present work, a simply supported vertical beam whose upper end is restrained and carries a concentrated mass and the lower end is subjected to a harmonic excitation force is considered. The large curvature deflection theory gives a partial differential equation for its lateral vibration. This equation has non-linear terms due to large curvature and longitudinal inertia of the concentrated mass and the beam elements. Here two mode approximation of the lateral vibration of the beam is considered and the governing partial differential equation of motion is reduced to two non-linear ordinary differential equations. After linearising these equations the instability boundaries are found out making use of Hills determinants. Taking one mode approximation of lateral vibration of the beam the instability boundaries and the effects of damping, mass

ratio and stiffness ratio on it are found out and hence, the result obtained by Gurgoze is reproduced. Taking one mode approximation the backbone curves are found out by method of harmonic balance, method of slowly varying parameters and method of multiple scales.

Chapter 2 deals with the formulation of the system i.e. to derive the equation of motion of the system. Chapter 3 deals with the instability boundaries which are found out by the method of Hills determinants . These boundaries are found out for both one mode approximation and two mode approximation. Chapters 4-6 deal with the method of hormonic balance, the method of multiple scales and method of solwly varying phase and amplitude to find out the steady state response. The data of reference [14] is used to get the backbone curves and these curves are plotted. The effects of mass ratio, stiffness ratio and excitation amplitudes(non-dimensional) on the response curve are found out. And the results from all these three methods are compared with each other. In fact the results from method of harmonic balance are reproduced as it was done by Gurgoze [14].

CHAPTER 2

FORMULATION

A simply supported vertical beam of uniform cross-section, carrying a concentrated mass at one end and subjected to a periodic axial displacement excitation at the other end is considered, Fig.1. The effects of shear deformation and rotary inertia of the beam are neglected as beam-thickness is small in comparison to the length. Bernoulli-Euler theory is used for large lateral deflection of the beam and as such the coordinate system chosen is such that the distance of any section from the lower hinge is considered along the arc of the elastic curve and the lower hinge point is the origin. To construct the Bernoulli-Euler equation, the moment of the spring force, longitudinal inertia forces of the concentrated mass and beam elements and lateral inertia force of the beam element are considered in the expression for bending moment at a cross section. These moments are calculated separately and are put in the Euler equation; differentiating the resulting equation twice with respect to the arc coordinate one gets the equation of motion which is a non-linear partial differential equation (P.D.E.). Then one mode and two mode approximation for its lateral vibration are considered and using Galerkin's method, the governing P.D.E. is converted into non-linear

ordinary differential equation(s) (O.D.E.) which is/ are subsequently normalised to the standard form of Mathieu equation(s).

2.1 DESCRIPTION OF THE SYSTEM:

In an internal combustion engine, the inlet and exhaust valves are operated by a cam system through engine-valve mechanism. The can lifts a cam follower connected to push rod. The push rod actuates a lever, called a rocker arm, which depresses the valve stem and opens the valve. A typical valve actuating system for an air-cooled engine is shown in Fig. 2.

This system is modelled, Fig. 1, such that the beam corresponds to push rod and mass M to the reduced mass of the rocker arm, valve and valve spring. The spring represents the equivalent stiffness of the valve spring and g(t) corresponds to the harmonic motion of the lower end of the push rod which is driven by a cam. The following assumptions are made:

(i) The thickness of the beam is small in comparison with the length of the beam. Therefore, the effect s of shear deformation and rotatory inertia are neglected.

(ii) The extensional deformation of the beam is negligible .

2.2 EQUATION OF MOTION:

From the Bernoulli-Euler theory, the bending moment M_b at any section s is given, in the coordinate system shown in Fig. 1, by

$$M_b = EI\, v_{,ss}(1-v^2_{,s})^{-\frac{1}{2}} \qquad \ldots\ldots (2.1)$$

where EI is the bending rigidity, v is the lateral displacement and subscript comma and s denotes differentiation with respect to the arc length S along the elastic curve.
For the parametrically excited beam shown in Fig.1, the bending moment at any section s can be written as

$$M_b = -F(t)v - M_1 + M_2 - M_3 \qquad \ldots\ldots (2.2)$$

where F(t) = sum of the spring and inertia forces excerted on the point mass.

M_1 = bending moment at section s due to lateral inertia force

M_2 = bending moment at section s due to longitudinal inertia force

and M_3 = bending moment at section s due to damping force.

The spring force at the upper end of the beam consists of preload on the spring F_o and the spring force due to the displacement of the spring which is caused by ground excitation. To calculate the inertia force at the

concentrated mass, the displacement of the concentrated mass for any instant t is found out and it is differentiated twice with respect to time to get its acceleration. Therefore, the force F(t) is given by

$$F(t) = F_o + [g(t) - (L - \int_o^L \cos \phi \; d\beta)] \, K_f + [g_{,tt} - \int_o^L (\cos \phi . \phi^2_{,t} + \sin\phi \, . \, \phi_{,tt}) \, d\beta] \, M \qquad \text{..... (2.3)}$$

where F_o = preload on the spring,

K_f = spring stiffness and its coefficient is the displacement of upper hinge point of the beam,

M = concentrated mass and its coefficient is the acceleration of upper hinge of the beam,

L = length of the beam,

β = arc length along the elastic curve

and ϕ = angle between vertical axis and the tangent to the elastic curve.

To find out the bending moment at any section S due to the lateral inertia force, one has to consider β along the beam which ranges from s to L. Then in this range one considers an element and finds out the lateral inertia of that element. To find out the total moment integrate the moment of the lateral inertia force over the entire range of β . Therefore

$$M_1 = \int_s^L \mu\, v_{,tt}\left(\int_s^\beta \cos\phi\, d\eta\right) d\beta - R\int_s^L \cos\phi\, d\beta \qquad \ldots. \quad (2.4)$$

where μ = beam mass per unit length

and R = reaction force at the upper end of the beam.See equation(2.7).

Similarly, to find out the bending moment at section s due to longitudinal inertia force one considers β along the beam which ranges from s to L and is given by

$$M_2 = \int_s^L \mu\left[g_{,tt} + \left(\beta - \int_o^\beta \cos\phi\, d\eta\right)_{,tt}\right]\left(\int_s^\beta \sin\phi\, d\eta\right) d\beta \qquad \ldots \quad (2.5)$$

where

η = arc coordinate along the elastic curve which ranges from S to β

and $\int_s^\beta \sin\phi\, d\eta$ = distance of the element from the vertical axis.

Similarly,the bending moments M_3 due to damping force distributed over s to L is given by

$$M_3 = \int_s^L c\, v_{,t}\left(\int_s^\beta \cos\phi\, d\eta\right) d\beta \qquad \ldots \quad (2.6)$$

where c = viscous damping coefficient,

η = arc coordinate considered along the elastic curve which ranges from s to β

and $\int_s^\beta \cos\phi\, d\eta$ = axial distance of the element from the section s considered.

To find out the horizontal reaction force R at the top end of the beam, take moment about the lower hinge

point of all the inertia forces and the damping force so as to obtain

$$R = [M_{11}(0) - M_2(0) + M_3(0)] / \int_0^L \cos\phi \, d\beta \qquad \ldots. (2.7)$$

where the first term in the expression for M_1 has been denoted as M_{11}.

Now it is sufficient to express the terms ϕ, $\phi_{,t}$, $\phi_{,tt}$, $\sin\phi$ and $\cos\phi$ by their power series approximations because only upto third order non-linear terms are to be retained in the governing differential equation and hence

$$\phi = v_{,\beta}$$

$$\phi_{,t} = v_{,\beta t}$$

$$\phi_{,tt} = v_{,\beta tt}$$

$$\sin\phi = \phi = v_{,\beta}$$

$$\cos\phi = 1 - \frac{\phi^2}{2} = 1 - \frac{1}{2} v^2_{,\beta}$$

Substituting these expressions into equations (2.3), (2.4), (2.5), (2.6) and (2.7), one obtains

$$F(t) = F_o + [g(t) - L + \int_0^L (1 - \frac{1}{2} v^2_{,\beta}) \, d\beta \,] K_f$$

$$+ [\, g_{,tt} - \int_0^L v^2_{,\beta t} \, d\beta - \int_0^L v_{,\beta} v_{,\beta tt} \, d\beta] M \qquad \ldots (2.8)$$

$$M_1 = \int_s^L \mu v_{,tt} (\int_s^\beta (1 - \frac{1}{2} v^2_{,\eta}) \, d\eta) \, d\beta$$

$$- R \int_s^L (1 - \frac{1}{2} v^2_{,\beta}) \, d\beta \qquad \ldots (2.9)$$

$$M_2 = \int_s^L \mu \, [g_{,tt} + \int_o^\beta (v^2_{,\eta t} + v_{,\eta}\, v_{,\eta tt})\, d\eta] \cdot (\int_s^\beta v_{,\eta} d\eta)\, d\beta \quad \ldots. (2.10)$$

$$M_3 = \int_s^L cv_{,t}\, [\int_s^\beta (1 - \tfrac{1}{2} v_{,\eta})\, d\eta]\, d\beta \quad \ldots. (2.11)$$

$$R = \frac{1}{L_p} (\int_o^L \mu v_{,tt}\, [\int_o^\beta (1 - \tfrac{1}{2} v^2_{,\eta})\, d\eta]\, d\beta - \int_o^L \mu\, [g_{,tt} + \int_o^\beta (v^2_{,\eta t} + v_{,\eta} v_{,\eta tt})\, d\eta] \cdot (\int_o^\beta v_{,\eta}\, d\eta)\, d\beta) + \frac{1}{L_p} (\int_o^L cv_{,t}\, [\int_o^\beta (1 - \tfrac{1}{2} v^2_{,\eta})\, d\eta]\, d\beta) \quad \ldots..(2.12)$$

where $$L_p = \int_o^L (1 - \tfrac{1}{2} v^2_{,\beta})\, d\beta \quad \ldots..(2.13)$$

= projected length of the elastic curve on the vertical axis.

Substituting the expressions (2.8) - (2.12) in equation (2.2) subject the equation (.2.13) and subsequently in the equation (2.1) and differentiating the resulting equation twice with respect to s one gets the following partial differential equation which contains all the principal non-linear terms upto the third order [See Appendix -I for details].

$$EI\, [v_{,ssss} + \tfrac{1}{2} v_{,ssss} v^2_{,s} + 3\, v_{,s}\, v_{,ss}\, v_{,sss} + v^3_{,ss}]$$

$$+ v_{,s}\, v_{,ss} \int_s^L cv_{,t} d\beta + cv_{,t} (1 - \tfrac{1}{2} v^2_{,s}) + F(t)\, v_{,ss}$$

$$+ v_{,s} v_{,ss} \int_s^L \mu v_{,tt} \, d\beta + \mu v_{,tt} \left(1 - \frac{1}{2} v_{,s}^2\right)$$

$$- v_{,s} v_{,ss} R + \mu v_{,ss} \int_s^L \left[g_{,tt} + \int_0^\beta (v_{,\eta t}^2 + v_{,\eta} v_{,\eta tt}) d\eta \right] d\beta$$

$$- \mu v_{,s} \left[g_{,tt} + \int_0^s (v_{,\eta t}^2 + v_{,\eta} v_{,\eta tt}) \, d\eta \right] = 0 \qquad \text{....(2.14)}$$

The boundary conditions for the simply supported beam are

$$v = v_{,ss} = 0 \text{ at } s = 0 \text{ and } s = L \qquad \text{....(2.15)}$$

A solution to the differential equation (2.14) could be sought in the form similar to that of lateral vibration of beam with small amplitude which is given by

$$v(s,t) = \sum_{n=1}^{\infty} B_n (t) \; \text{Sin} \; \frac{n\pi s}{L} \qquad \text{....(2.16)}$$

Experimental evidence shows that the first spatial mode is dominant. But here upto second mode is considered so as to investigate the effect of higher mode on the instability boundaries of the beam.

2.3 ONE MODE APPROXIMATION:

One mode approximation for the equation (2.14) is

$$v(s,t) = B(t) \; \text{Sin} \; \frac{\pi s}{L} \qquad \text{... (2.17)}$$

where B(t) represents a time dependent coefficient to be determined from Galerkin's method. In this method, one puts the approximate solution in the left hand side of the

equation (2.14) and the error function is orthogonalised with respect to the trial function . For one mode approximation the error function is orthogonalised with respect to Sin $\frac{\pi s}{L}$ only to obtain the following O.D.E. which has been transformed to nondimensional equation.

$$\bar{B}_{,zz} + \frac{c}{\mu\omega} \bar{B}_{,z} + [\frac{1}{\omega^2} (\omega_{01}^2 - K_{f2} \bar{g} - M_{22} \bar{\sigma}_{,zz}] \bar{B}$$

$$+ \frac{1}{\omega^2}[(\frac{\pi^2}{8}) \omega_o^2 + K_{f4}] \bar{B}^3 + M_{46} \bar{B} \bar{B}_{,z}^2$$

$$- \frac{3\pi^2}{8} \frac{c}{\mu\omega} \bar{B}^2 \bar{B}_{,z} + M_{44} \bar{B}^2 \bar{B}_{,zz} = 0 \qquad \text{...(2.18)}$$

where, ω is the angular frequency of the prescribed motion,

$z = \omega t$, $\bar{B} = \frac{B}{L}$, $\bar{g} = \frac{g}{L}$,

$\omega_o^2 = \frac{\pi^4 EI}{\mu L^4}$, $\omega_{o1}^2 = \omega_o^2 - \frac{\pi^2 F_o}{\mu L^2}$,

$K_{f2} = \frac{\pi^2 K_f}{\mu L}$, $K_{f4} = \frac{\pi^4 K_f}{4\mu L}$, $M_{22} = \pi^2 [\frac{M}{\mu L} + \frac{1}{2}]$,

$M_{46} = \frac{\pi^2}{2} [\pi^2 (\frac{M}{\mu L}) - \frac{\pi^2}{3} + \frac{3}{8}]$

and $M_{44} = \frac{\pi^2}{2} [\pi^2 \frac{M}{\mu L} - \frac{\pi^2}{3} - \frac{3}{8}]$ .....(2.19)

Here, ω_o = fundamental bending eigen frequency of the beam

and ω_{o1} = fundamental bending eigen frequency of the same beam when compressed by the axial force F_o.

Depending upon the cam profile there are different cam-displacement curves which are periodic in nature. Also

in cam-follower mechanisms there can be loss of contact between the cam and the follower which depend upon its angular velocity and the com follower geometry. However, here it is assumed that the displacement excitation is a simple harmonic motion and there is no such loss of contact during motion. As such, the displacement is given by

$$\bar{g}(z) = \epsilon\ (\ 1 - \mathrm{Cos}\ z) \qquad \ldots.\ (2.20)$$

Substituting (2.20) in equation (2.18), one gets

$$\bar{B}_{,zz} + \frac{c}{\mu\omega}\ \bar{B}_{,z} + [\frac{1}{\omega^2}\ (\ \omega_{o1}^2 - \epsilon\ K_{f2}) + \epsilon\ (\frac{K_{f2}}{\omega^2} - M_{22})\ \mathrm{Cos}\ z]\bar{B}$$

$$+ \frac{1}{\omega^2}\ [\ (\frac{\pi^2}{8})\ \omega_o^2\ + K_{f4}]\ \bar{B}^{\ 3} + M_{46}\ \bar{B}\ \ \bar{B}^2_{,z}$$

$$- \ \frac{3\pi^2}{8}\ \ \frac{c}{\mu\omega}\ \bar{B}^2\ \bar{B}_{,z} + M_{44}\ \ \bar{B}^2\ \bar{B}_{,zz} = 0 \qquad \ldots.\ (2.21)$$

Equation (2.21) has seven terms and starting from the first these represent inertia, linear damping, linear stiffness and parametric excitation, the effect of non-linear curvature on the non- linear elasticity (geometric non-linerity only), the non-linear longitudinal inertial force of the beam itself, the non-linear character of damping and the non-linear inertia force of the concentrated mass, respectively.

2.4 TWO MODE APPROXIMATION :

For two mode approximation of the response of the system, express v(s,t) as

$$v(s,t) = B_1(t)\ \mathrm{Sin}\ \frac{\pi s}{L} + B_2(t)\ \mathrm{Sin}\ \frac{2\pi s}{L} \qquad \ldots (2.22)$$

where, the unknown functions $B_1(t)$ and $B_2(t)$ are evaluated by using Galerkin's method as discussed earlier. To this end B_1, B_2 and $g(t)$ are normalised with respect to L and the following two equations are obtained

$$\frac{\mu L^2}{2}\ \ddot{\bar{B}}_1 + \frac{cL^2}{2}\ \dot{\bar{B}}_1 + \left(\frac{\pi^4 EI}{2L^2} - \frac{\pi^2}{2}(F_o+\bar{g}(t)K_f\ L+\bar{g}_{,tt}\ ML) - \frac{\mu\pi^2}{4}\ \bar{g}_{,tt}L^2\right) \bar{B}_1 - \frac{20}{9}\ \mu L^2\bar{g}_{,tt}\ \bar{B}2$$

$$+ \psi_{11}(\bar{B}_1, \dot{\bar{B}}_1, \ddot{\bar{B}}_1) + \psi_{12}(\bar{B}_2, \dot{\bar{B}}_2, \ddot{\bar{B}}_2)$$

$$+ \psi_{13}(\bar{B}_1, \bar{B}_2, \dot{\bar{B}}_1, \dot{\bar{B}}_2, \ddot{\bar{B}}_1, \ddot{\bar{B}}_2) = 0 \qquad \ldots (2.23)$$

and

$$\frac{\mu L^2}{2}\ \ddot{\bar{B}}_2 + \left[\frac{8\pi^4 EI}{L^2} - 2\pi^2(F_o+\bar{g}(t)\ K_f L + \bar{g}_{,tt}\ ML)\right.$$

$$\left. - \mu\pi^2 L^2\ \bar{g}_{,tt}\right] \bar{B}_2 + \frac{cL^2\dot{\bar{B}}_2}{2} - \frac{20\mu}{9}\ \bar{g}_{,tt}\ L^2\ \bar{B}_1$$

$$+ \psi_{21}(\bar{B}_1, \dot{\bar{B}}_1, \ddot{\bar{B}}_1) + \psi_{22}(\bar{B}_2, \dot{\bar{B}}_2\ \ddot{\bar{B}}_2)$$

$$+ \psi_{23}(\bar{B}_1, \bar{B}_2, \dot{\bar{B}}_1, \dot{\bar{B}}_2, \ddot{\bar{B}}_1, \ddot{\bar{B}}_2) = 0 \qquad \ldots (2.24)$$

where

$$\bar{B}_1(t) = \frac{B_1(t)}{L},$$

$$\bar{B}_2(t) = \frac{B_2(t)}{L},$$

$$\bar{g}(t) = \frac{g(t)}{L}$$

and the functions ψ_{ij} are given by

$$\psi_{11}(\bar{B}_1, \dot{\bar{B}}_1, \ddot{\bar{B}}_1) = \left(\frac{\pi^6 EI}{16L^2} + \frac{\pi^4 K_f L}{8}\right) \bar{B}_1^3$$

$$+ \frac{\pi^4 ML}{4} \bar{B}_1 (\dot{\bar{B}}_1^2 + \bar{B}_1 \ddot{\bar{B}}_1) - \frac{3\pi^2 cL^2}{16} \bar{B}_1^2 \dot{\bar{B}}_1$$

$$- \frac{3\mu\pi^2 L^2}{16} \ddot{\bar{B}}_1 \bar{B}_1^2 + \frac{3}{32} \mu\pi^2 L^2 \bar{B}_1 (\dot{\bar{B}}_1^2 + \bar{B}_1 \ddot{\bar{B}}_1)$$

$$- \frac{\mu\pi^4 L^2}{12} \bar{B}_1 (\dot{\bar{B}}_1^2 + \bar{B}_1 \ddot{\bar{B}}_1),$$

$$\psi_{12}(\bar{B}_2, \dot{\bar{B}}_2, \ddot{\bar{B}}_2) = -\frac{20}{9} \mu\pi^2 L^2 \bar{B}_2 (\dot{\bar{B}}_2^2 + \bar{B}_2 \ddot{\bar{B}}_2),$$

$$\psi_{13}(\bar{B}_1, \bar{B}_2, \dot{\bar{B}}_1, \dot{\bar{B}}_2, \ddot{\bar{B}}_1, \ddot{\bar{B}}_2)$$

$$= \left(\frac{\pi^6 EI}{2L^2} + \frac{\pi^4 K_f L}{2}\right) \bar{B}_1 - \frac{c\pi^2 L^2}{2} \left(\frac{1}{2} \bar{B}_1 \bar{B}_2 \dot{\bar{B}}_2 + \bar{B}_2^2 \dot{\bar{B}}_1\right)$$

$$+ \pi^4 ML \bar{B}_1 (\dot{\bar{B}}_2^2 + \bar{B}_2 \ddot{\bar{B}}_2) + \mu\pi^2 L^2 \Big[-\ddot{\bar{B}}_1 \left(\frac{19}{18} \bar{B}_2^2 + \frac{10}{9} \bar{B}_1 \bar{B}_2\right)$$

$$- \ddot{\bar{B}}_2 \left(\frac{5}{9} \bar{B}_1^2 + \frac{11}{36} \bar{B}_1 \bar{B}_2\right) - \frac{5}{9} \dot{\bar{B}}_1^2 \bar{B}_2 + \frac{1}{2} \bar{B}_1 \dot{\bar{B}}_2^2$$

$$- \frac{10}{9} \dot{\bar{B}}_1 \dot{\bar{B}}_2 (\bar{B}_1 + \bar{B}_2) \Big] - \frac{\mu\pi^2 L^2}{2} \bar{g}_{,tt} \bar{B}_1^2 \bar{B}_2$$

$$- \mu\pi^4 L^2 \left(\frac{1}{3} + \frac{3}{16\pi^2}\right) \bar{B}_1 (\dot{\bar{B}}_2^2 + \bar{B}_2 \ddot{\bar{B}}_2),$$

$$\psi_{21}(\bar{B}_1, \dot{\bar{B}}_1, \ddot{\bar{B}}_1) = -\frac{5}{9} \mu\pi^2 L^2 \bar{B}_1 (\dot{\bar{B}}_1^2 + \bar{B}_1 \ddot{\bar{B}}_1)$$

$$- \frac{\mu\pi^2 L^2}{4} \bar{g}_{,tt} \bar{B}_1^3,$$

$$\psi_{22}\ (\bar{B}_2, \dot{\bar{B}}_2, \ddot{\bar{B}}_2) = \left(\frac{4\pi^6 EI}{L^2} + 2\pi^4 K_f L\right) \bar{B}_2^3$$

$$+ \quad \left[4\pi^4 ML + \left(-\frac{1}{2} + \frac{8\pi^2}{3}\right) \mu\pi^2 L^2\right] \bar{B}_2^2\ \ddot{\bar{B}}_2 - \frac{3c\pi^2 L^2}{4}\ \bar{B}_2^2\ \dot{\bar{B}}_2$$

$$+ \quad \bar{B}_2\ \dot{\bar{B}}_2^2\ \left[4\pi^2 ML + \left(\frac{7}{8} - \frac{8\pi^2}{3}\right) \mu\pi^2 L^2\right],$$

and $\psi_{23}(\bar{B}_1, \bar{B}_2, \dot{\bar{B}}_1, \dot{\bar{B}}_2\ \ddot{\bar{B}}_1 , \ddot{\bar{B}}_2)$

$$= \quad \left(\frac{2\pi^6 EI}{L^2} + \frac{\pi^4 K_f L}{2} \right) \bar{B}_1^2\ \bar{B}_2 - c\pi^2 L^2\ \bar{B}_1\ \bar{B}_2\ \dot{\bar{B}}_1$$

$$+ \quad \left[\pi^4 ML + \left(\frac{5}{16} - \frac{\pi^2}{3}\right) \mu\pi^2 L^2\right] \dot{\bar{B}}_1{}^2\ \bar{B}_2$$

$$+ \quad \left[\pi^4 ML - \left(\frac{\pi^2}{3} + \frac{179}{144}\right) \mu\pi^2 L^2\right] \bar{B}_1 \bar{B}_2\ \ddot{\bar{B}}_1$$

$$- \quad \frac{179}{144}\ \mu\pi^2 L^2\ \bar{B}_1^2\ \ddot{\bar{B}}_2 - \frac{20}{9}\ \mu\pi^2 L^2\ \bar{B}_1\ \dot{\bar{B}}_2^2$$

$$- \quad \frac{40}{9}\ \mu\pi^2 L^2\ \bar{B}_1 \bar{B}_2\ \ddot{\bar{B}}_2 - \frac{43}{36}\ \mu\pi^2 L^2\ \bar{B}_1\ \dot{\bar{B}}_1\ \dot{\bar{B}}_2$$

$$- \quad \frac{20}{9}\ \mu\pi^2 L^2\ \ddot{\bar{B}}_1\ \bar{B}_2^2 - \frac{40}{9}\ \mu\pi^2 L^2\ \dot{\bar{B}}_1 \dot{\bar{B}}_2\ \bar{B}_2$$

$$- \quad \frac{c\pi^2 L^2}{8}\ \bar{B}_1^2\ \dot{\bar{B}}_2$$

Now linearising the equations (2.23) and (2.24) and introducing the new time scale $z = \omega t$ the following equations are obtained.

$$q^2\ \bar{B}_{1,zz} + \varepsilon\ c_* q\ \bar{B}_{1,z} + \left(\frac{1}{4} - \frac{\varepsilon K_{f2}}{4\omega_{o1}{}^2}\right) \bar{B}_1$$

$$+ \varepsilon \left(\frac{K_{f2}}{4\omega_{o1}{}^2} - q^2\ M_{22}\right) \text{Cos } z\ \bar{B}_1 - \frac{40\ \varepsilon}{9}\ q^2\ \text{Cos } z\ \bar{B}_2 = 0$$

and

$$q^2\bar{B}_{2,zz} + \varepsilon\, c_* \, q\, \bar{B}_{2,z} + \left(1 + \frac{3\omega_o^2}{\omega_{o1}^2} - \frac{\varepsilon K_{f2}}{\omega_{o1}^2}\right)\bar{B}_2$$

$$+ \varepsilon \left(\frac{K_{f2}}{\omega_{o1}^2} - 4q^2 M_{22}\right) \text{Cos } z\, \bar{B}_2 - \frac{40\,\varepsilon}{9}\, q^2 \text{ Cos } z\, \bar{B}_1 = 0$$

..... (2.25)

where

$$c_* = \frac{c}{2\mu\varepsilon\omega_{o1}}$$

and

$$q = \frac{\omega}{2\omega_{o1}}$$

These are two coupled linear equations with periodic coefficients, refered to as coupled Mathieu equation s.

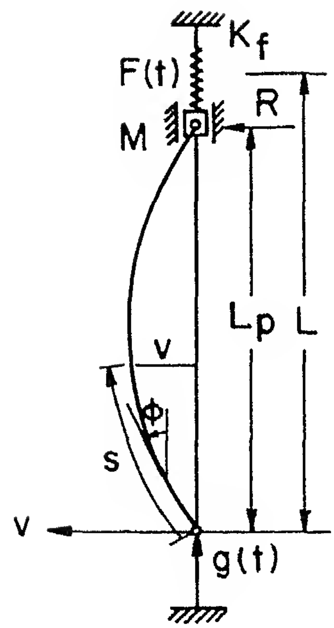

Fig. 1 Restrained beam with end mass

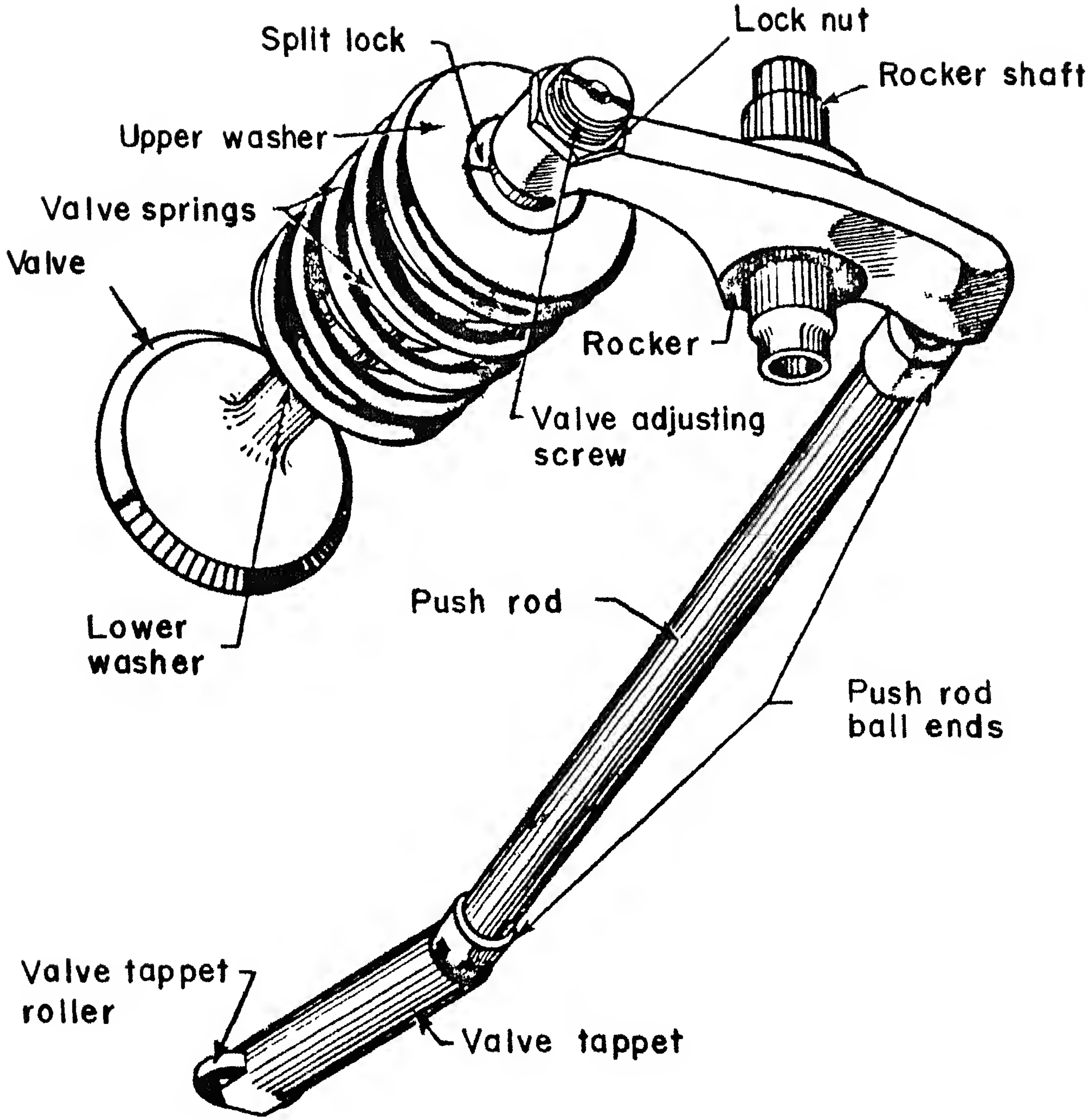

Fig.2 Overhead-valve, rocker arm and push-rod assembly.

CHAPTER 3

INSTABILITY ANALYSIS

The present chapter deals with the stable and unstable regions in the ε, q plane for two coupled Mathieu equations for two mode approximation of lateral vibration of the beam. The parameters ε and q refer to non-dimensional excitation amplitude and the ratio of excitation frequency to the bending eigen frequency of the beam respectively. The transition curves $\varepsilon = \varepsilon\,(q, M_*, K_{f*}, \delta_d)$ separating the stable and unstable regions for different parameters like mass ratio M_*, stiffness ratio K_{f*} and damping δ_d are of prime interest. The Mathieu equations have 2π - periodic coefficients, therefore, from Floquet theory it is concluded that for such values of ε and q these Mathieu equations possess a periodic solution having either period 2π or period 4π. Since these periodic solutions $B_1(t)$ and $B_2(t)$ are regular for all values of time it follows that these can be expressed by appropriate Fourier series of 2π and 4π periods. Substituting these in the DE of $B_1(t)$ and $B_2(t)$ and using method of harmonic balance, one gets two sets of homogeneous linear algebraic equations. As all the Fourier constants can not vanish simultaneously the coefficient matrix must be singular. This condition leads to two equations between ε and q in implicit form. From these one can fined transition values of ε and q using a Nag-subroutine CO5AJF [17]

which attempts to locate the zeros of a continuous function by a continuation method using a secant iteration. For the function, given in the form of a determinant, obtained in the present case, the determinant is calculated by use of another Nag-subroutine FO3AAF [18] which utilizes Crouts factorisation method.

3.1 ANALYSIS OF INSTABILITY REGIONS:

A standard Mathieu equation which governs a wide variety of problems in mathematical physics is

$$\frac{d^2w}{dz^2} + (\delta + \epsilon \operatorname{Cos} z)\, w = 0$$

For the system considered the corresponding equations for two mode approximation are

$$q^2 \bar{B}_{1,zz} + \epsilon c_* q \bar{B}_{1,z} + \left(\frac{1}{4} - \frac{\epsilon K_{f2}}{4\omega_{o1}^2}\right)\bar{B}_1$$

$$+ \epsilon\left(\frac{K_{f2}}{4\omega_{o1}^2} - q^2 M_{22}\right) \operatorname{Cos} z\, \bar{B}_1 - \frac{40\epsilon}{9} q^2 \operatorname{Cos} z\, \bar{B}_2 = 0 \qquad \ldots\ldots (3.1)$$

and

$$q^2 \bar{B}_{2,zz} + \epsilon c_* q \bar{B}_{2,z} + \left(1 + \frac{3\omega_o^2}{\omega_{o1}^2} - \frac{\epsilon K_{f2}}{\omega_{o1}^2}\right) \bar{B}_2$$

$$+ \epsilon\left(\frac{K_{f2}}{\omega_{o1}^2} - 4q^2 M_{22}\right) \operatorname{Cos} z\, \bar{B}_2 - \frac{40\epsilon}{9} q^2 \operatorname{Cos} z\, \bar{B}_1 = 0 \qquad \ldots\ldots (3.2)$$

These are two coupled differential equations which are obtained after linearising the equations (2.23 and 2.24) obtained by applying Galerkin's method to the governing partial differential equation of motion with two mode approximation. As the coefficients in these equations have periodicity of 2π one has 2π periodic and also 4π periodic solutions of these two equations for obtaining the transition points from stability to instability.

(a) <u>For 2π Periodic Solutions:</u>

For 2π - periodic solutions one can write the solutions of $\bar{B}_1$ and $\bar{B}_2$ by the following Fourier series

$$\bar{B}_1 = a_o + \sum_{n=1}^{N} (a_n \text{ Cos } n\, z + b_n \text{ Sin } n\, z)$$
$$\bar{B}_2 = a_o' + \sum_{n=1}^{N} (a_n' \text{ Cos } n\, z + b_n' \text{ Sin } n\, z) \qquad \ldots.(3.3)$$

Substituting the equations (3.3) for $\bar{B}_1(t)$ and $\bar{B}_2$ (t) into equations (3.1) and (3.2) and comparing the harmonics, one gets recurrence relations for a_n, b_n, a_n' and b_n'. In the present analysis, N is taken as 3. In fact if N is increased, one can get more accurate results but that is at the expense of higher computation time. Having done the harmonic balance one gets a system of simultaneous, homogeneous algebraic equations where the unknowns are the Fourier constants. Write this system of equations into a matrix form

$$[CM2]_{14X14}[\, a_o\ a_1\ b_1\ a_2\ b_2\ a_3\ b_3\ a'_o\ a_1'\ b_1'\ a_2'\ b_2'\ a_3'\ b_3']^T = [O]_{1X14} \qquad \ldots..(3.4)$$

where CM2 is the coefficient matrix for 2π periodic solution given on page no 30 . Since all the Fourier coefficients can not vanish simultaneously for the non-trivial solution, the coefficient matrix has to be singular i.e.

$$\det [CM2] = 0 \qquad \ldots. (3.5)$$

This is an explicit equation in ε and q. To get the transition values of ε and q set the value of ε equal to a prescribed value and solve equation (3.5) for q. For $\varepsilon = 0$, one gets the transition points on q axis, called as bifurcation points wherefrom the instability boundaries bifurcate. For this the solution procedure is as follows:

Start with $\varepsilon = 0$ and any one of the values of q obtained therein, increase ε by some amount and look for solution of q from the matrix equation (3.5) with an initial guess of q using the Nag subroutine CO5AJF [17]. (It attempts to locate a zero of a continuous function by a continuation method by using a secant iteration). To locate the transition curves in the neighbourhood of a specific bifurcation point $q_{\varepsilon=0}$, search for values of q for a prescribed ε by initial trial values of $q_{\varepsilon=0} - \Delta q$ and $q_{\varepsilon=0} + \Delta q$. Continue increasing ε and obtain the correspon-ding values of q in the neighbourhood of $q_{\varepsilon=0}$ by adopting the current value of q as the initial guess. Repeat this procedure for all the bifurcation points $q_{\varepsilon=0}(i), i=0,1,2\ldots$

(b) For 4π Periodic Solutions:

For 4π periodic solution of equations (3.1) and (3.2), one writes the following Fourier series solutions

$$\bar{B}_1 = \sum_{n=1,3,\ldots}^{N} \left(a_n \cos \frac{nz}{2} + b_n \sin \frac{nz}{2}\right)$$
$$\bar{B}_2 = \sum_{n=1,3,\ldots}^{N} \left(a_n' \cos \frac{nz}{2} + b_n' \sin \frac{nz}{2}\right) \qquad \ldots (3.6)$$

Substituting (3.6) with $N = 5$ in equations (3.1) and (3.2) and comparing the harmonics the matrix equation obtained is similar to that of 2π- periodic solution and is given by $[CM4]_{12X12}[a_1\ b_1\ a_3 b_3 a_5 b_5 a_1' b_1' a_3' b_3' a_5' b_5']^T$

$$= [0]_{1X12} \qquad \ldots (3.7)$$

where [CM4] is the coefficient matrix for 4π periodic solution given on page no 31 . This matrix is singular as equation (3.7) is a system of homogeneous simultaneous algebraic equations in Fourier coefficients which do not all vanish simultaneously for non- trivial solution. So

$$\det [CM4] = 0 \qquad \ldots (3.8)$$

It is an explicit equation in ϵ and q describing the transition values ϵ and q for 4π periodic solution of (3.1) and (3.2). The method for obtaining the transition values has been discussed earlier.

3.2 RESULTS:

Firstly, the principal parametric instability region is found out. To do this, one mode (the first one) approximation of lateral vibration is made i.e. putting N=I in the first equation of (3.6) and neglecting the second equation of (3.6). Therefore, a set of homogeneous simultaneous algebraic equations is a_1 and b_1 is obtained i.e. in equation (3.8) determinant [CM4] is of order 2X2 from which E versus q graph is plotted for the following values of the parameters $\delta_d = 0$, $M_* = 0.1$, $K_{f*} = 50$ and $\frac{F_o}{P_{cr}} = 0$. The results obtained (Fig. 3a) are comparable with those obtained by Gurgoze [14]. For the same one mode approximation and taking three terms of the Fourier series, the determinant [CM4] in equation (3.8) becomes 6X6. Again the principal parametric instability region is found out as shown in Fig.3(a). Here a slight change in the instability boundaries is noticed. The bifurcation curves have also been obtained for $\delta_d = 0.1$ by taking N=1 and N=5. It is observed that the instability boundary is lifted up for both the cases of N. This is similar to one reported by Gurgoze and Nayfeh.

Results for two mode approximation have also been obtained and are shown in Fig.3(b). The principal parametric instability region obtained is also compared with that of Gurgoze for one mode approximation.

$$
\begin{bmatrix}
\frac{1}{4}-\epsilon K_* & \frac{\epsilon}{2}(K_*-q^2M_{22}) & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{20\epsilon q^2}{9} & 0 & 0 & 0 & 0 & 0 \\
\epsilon(K_*-q^2M_{22}) & -q^2+\frac{1}{4}-\epsilon K_* & \epsilon C_* q & \frac{\epsilon}{2}(K_*-q^2M_{22}) & 0 & 0 & 0 & -\frac{40\epsilon q^2}{9} & 0 & 0 & -\frac{20\epsilon q^2}{9} & 0 & 0 & 0 \\
0 & -\epsilon C_* q & -q^2+\frac{1}{4}-\epsilon K_* & 0 & \frac{\epsilon}{2}(K_*-q^2M_{22}) & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{20\epsilon q^2}{9} & 0 & 0 \\
0 & \frac{\epsilon}{2}(K_*-q^2M_{22}) & 0 & -4q^2+\frac{1}{4}-\epsilon K_* & 2\epsilon C_* q & \frac{\epsilon}{2}(K_*-q^2M_{22}) & 0 & 0 & -\frac{20\epsilon q^2}{9} & 0 & 0 & 0 & -\frac{20\epsilon q^2}{9} & 0 \\
0 & 0 & \frac{\epsilon}{2}(K_*-q^2M_{22}) & -2\epsilon C_* q & -4q^2+\frac{1}{4}-\epsilon K_* & 0 & \frac{\epsilon}{2}(K_*-q^2M_{22}) & 0 & 0 & -\frac{20\epsilon q^2}{9} & 0 & 0 & 0 & -\frac{20\epsilon q^2}{9} \\
0 & 0 & 0 & \frac{\epsilon}{2}(K_*-q^2M_{22}) & 0 & -9q^2+\frac{1}{4}-\epsilon K_* & 3\epsilon C_* q & 0 & 0 & 0 & -\frac{20\epsilon q^2}{9} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{\epsilon}{2}(K_*-q^2M_{22}) & -3\epsilon C_* q & -9q^2+\frac{1}{4}-\epsilon K_* & 0 & 0 & 0 & 0 & -\frac{20\epsilon q^2}{9} & 0 & 0 \\
0 & -\frac{20\epsilon q^2}{9} & 0 & 0 & 0 & 0 & 0 & m-4\epsilon K_* & 2\epsilon(K_*-q^2M_{22}) & 0 & 0 & 0 & 0 & 0 \\
-\frac{40\epsilon q^2}{9} & 0 & 0 & -\frac{20\epsilon q^2}{9} & 0 & 0 & 0 & 4\epsilon(K_*-q^2M_{22}) & -q^2+m-4\epsilon K_* & \epsilon C_* q & 2\epsilon(K_*-q^2M_{22}) & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -\frac{20\epsilon q^2}{9} & 0 & 0 & 0 & -\epsilon C_* q & -q^2+m-4\epsilon K_* & 0 & 2\epsilon(K_*-q^2M_{22}) & 0 & 0 \\
0 & -\frac{20\epsilon q^2}{9} & 0 & 0 & 0 & -\frac{20\epsilon q^2}{9} & 0 & 0 & 2\epsilon(K_*-q^2M_{22}) & 0 & -4q^2+m-4\epsilon K_* & 2\epsilon C_* q & 2\epsilon(K_*-q^2M_{22}) & 0 \\
0 & 0 & -\frac{20\epsilon q^2}{9} & 0 & 0 & 0 & -\frac{20\epsilon q^2}{9} & 0 & 0 & 2\epsilon(K_*-q^2M_{22}) & -2\epsilon C_* q & -4q^2+m-4\epsilon K_* & 0 & 2\epsilon(K_*-q^2M_{22}) \\
0 & 0 & 0 & -\frac{20\epsilon q^2}{9} & 0 & 0 & 0 & 0 & 0 & 0 & 2\epsilon(K_*-q^2M_{22}) & 0 & -9q^2+m-4\epsilon K_* & 3\epsilon C_* q \\
0 & 0 & 0 & 0 & -\frac{20\epsilon q^2}{9} & 0 & 0 & 0 & 0 & 0 & 0 & 2\epsilon(K_*-q^2M_{22}) & -3\epsilon C_* q & -9q^2+m-4\epsilon K_*
\end{bmatrix}
$$

$$
\begin{bmatrix}
-\frac{q^2}{4}+\frac{1}{4}-\epsilon K_* +\frac{\epsilon}{2}(K_*-q^2M_{22}) & \frac{\epsilon C_* q}{2} & \frac{\epsilon}{2}(K_*-q^2M_{22}) & 0 & 0 & 0 & -\frac{20\epsilon q^2}{9} & 0 & -\frac{20\epsilon q^2}{9} & 0 & 0 & 0 \\
-\frac{\epsilon C_* q}{2} & -\frac{q^2}{4}+\frac{1}{4}-\epsilon K_* -\frac{\epsilon}{2}(K_*-q^2M_{22}) & 0 & \frac{\epsilon}{2}(K_*-q^2M_{22}) & 0 & 0 & 0 & \frac{20\epsilon q^2}{9} & 0 & -\frac{20\epsilon q^2}{9} & 0 & 0 \\
\frac{\epsilon}{2}(K_*-q^2M_{22}) & 0 & -\frac{9}{4}q^2+\frac{1}{4}-\epsilon K_* & \frac{3\epsilon C_* q}{2} & \frac{\epsilon}{2}(K_*-q^2M_{22}) & 0 & -\frac{20\epsilon q^2}{9} & 0 & 0 & 0 & -\frac{20\epsilon q^2}{9} & 0 \\
0 & \frac{\epsilon}{2}(K_*-q^2M_{22}) & -\frac{3\epsilon C_* q}{2} & -\frac{9}{4}q^2+\frac{1}{4}-\epsilon K_* & 0 & \frac{\epsilon}{2}(K_*-q^2M_{22}) & 0 & -\frac{20\epsilon q^2}{9} & 0 & 0 & 0 & -\frac{20\epsilon q^2}{9} \\
0 & 0 & \frac{\epsilon}{2}(K_*-q^2M_{22}) & 0 & -\frac{25}{4}q^2+\frac{1}{4}-\epsilon K_* & \frac{5\epsilon C_* q}{2} & 0 & 0 & -\frac{20\epsilon q^2}{9} & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{\epsilon}{2}(K_*-q^2M_{22}) & -\frac{5\epsilon C_* q}{2} & -\frac{25q^2}{4}+\frac{1}{4}-\epsilon K_* & 0 & 0 & 0 & -\frac{20\epsilon q^2}{9} & 0 & 0 \\
-\frac{20\epsilon q^2}{9} & 0 & -\frac{20\epsilon q^2}{9} & 0 & 0 & 0 & -\frac{q^2}{4}+m-4\epsilon K_* +2\epsilon(K_*-q^2M_{22}) & \frac{\epsilon C_* q}{2} & 2\epsilon(K_*-q^2M_{22}) & 0 & 0 & 0 \\
0 & \frac{20\epsilon q^2}{9} & 0 & -\frac{20\epsilon q^2}{9} & 0 & 0 & -\frac{\epsilon C_* q}{2} & -\frac{q^2}{4}+m-4\epsilon K_* -2\epsilon(K_*-q^2M_{22}) & 0 & 2\epsilon(K_*-q^2M_{22}) & 0 & 0 \\
-\frac{20\epsilon q^2}{9} & 0 & 0 & 0 & -\frac{20\epsilon q^2}{9} & 0 & 2\epsilon(K_*-q^2M_{22}) & 0 & -\frac{9q^2}{4}+m-4\epsilon K_* & \frac{3\epsilon C_* q}{2} & 2\epsilon(K_*-q^2M_{22}) & 0 \\
0 & -\frac{20\epsilon q^2}{9} & 0 & 0 & 0 & -\frac{20\epsilon q^2}{9} & 0 & 2\epsilon(K_*-q^2M_{22}) & -\frac{3\epsilon C_* q}{2} & -\frac{9q^2}{4}+m-4\epsilon K_* & 0 & 2\epsilon(K_*-q^2M_{22}) \\
0 & 0 & -\frac{20\epsilon q^2}{9} & 0 & 0 & 0 & 0 & 0 & 2\epsilon(K_*-q^2M_{22}) & 0 & -\frac{25}{4}q^2+m-4\epsilon K_* & \frac{5\epsilon C_* q}{2} \\
0 & 0 & 0 & -\frac{20\epsilon q^2}{9} & 0 & 0 & 0 & 0 & 0 & 2\epsilon(K_*-q^2M_{22}) & -\frac{5\epsilon C_* q}{2} & -\frac{25q^2}{4}+m-4\epsilon K_*
\end{bmatrix}
$$

Here $q = \frac{\omega}{2\omega_{01}}$, $C_* = \frac{c}{2\mu\epsilon\omega_{01}}$, $K_* = \frac{K_{f2}}{4\omega_{01}^2} = (1/4\pi^2)K_{fx}/(1-F_0/P_{cr})$, $K_{fx} = K_f/(EI/L^3)$,

$P_{..} = \pi^2 EI/L^2$ and $m = 1 + 3\omega_0^2/\omega_{01}^2$.

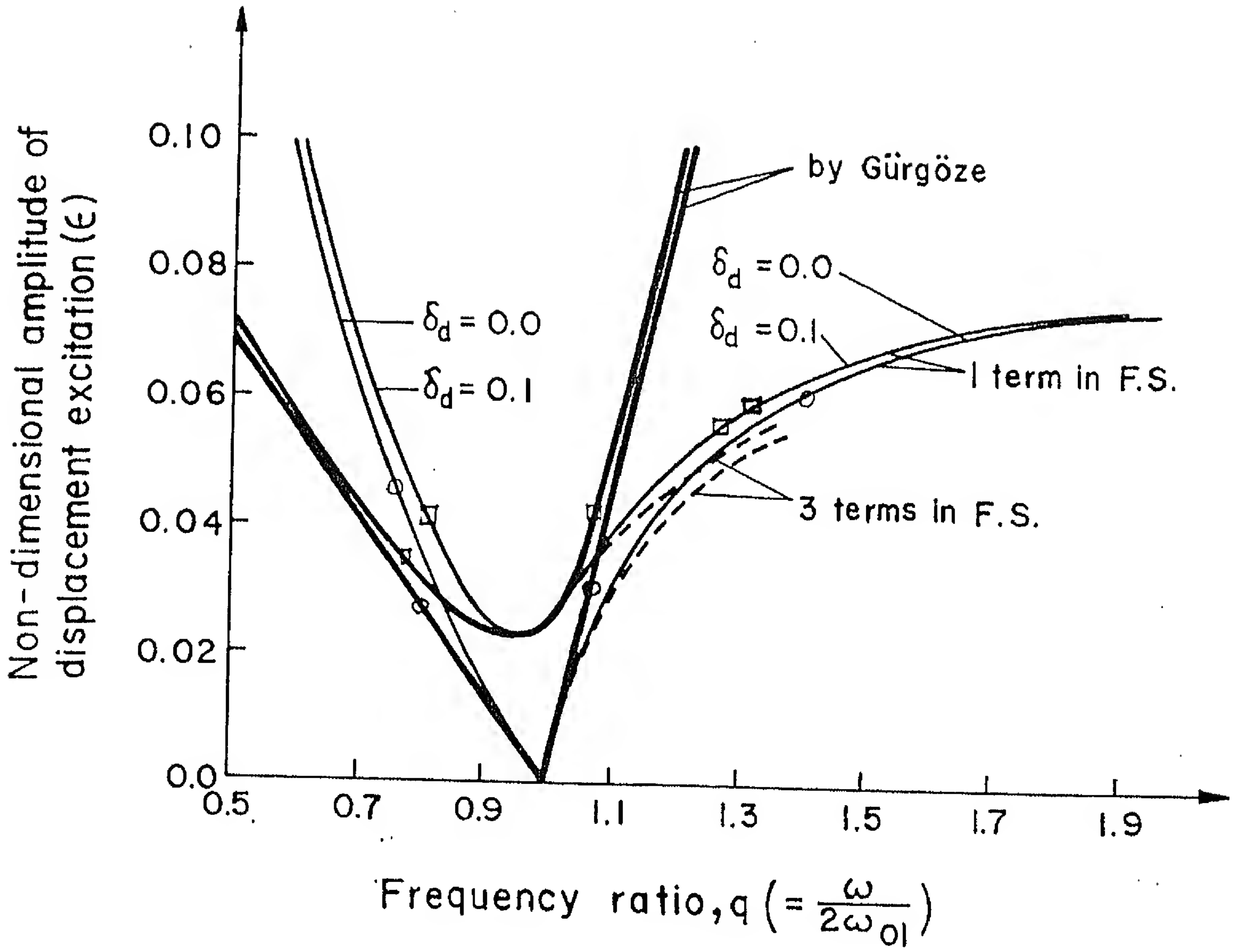

Fig. 3a Principal parametric instability region for one mode approximation (compared with that by Gürgöze) with $M_* = 0.1$, $K_{f*} = 50$ and $\frac{F_0}{P_{cr}} = 0.0$

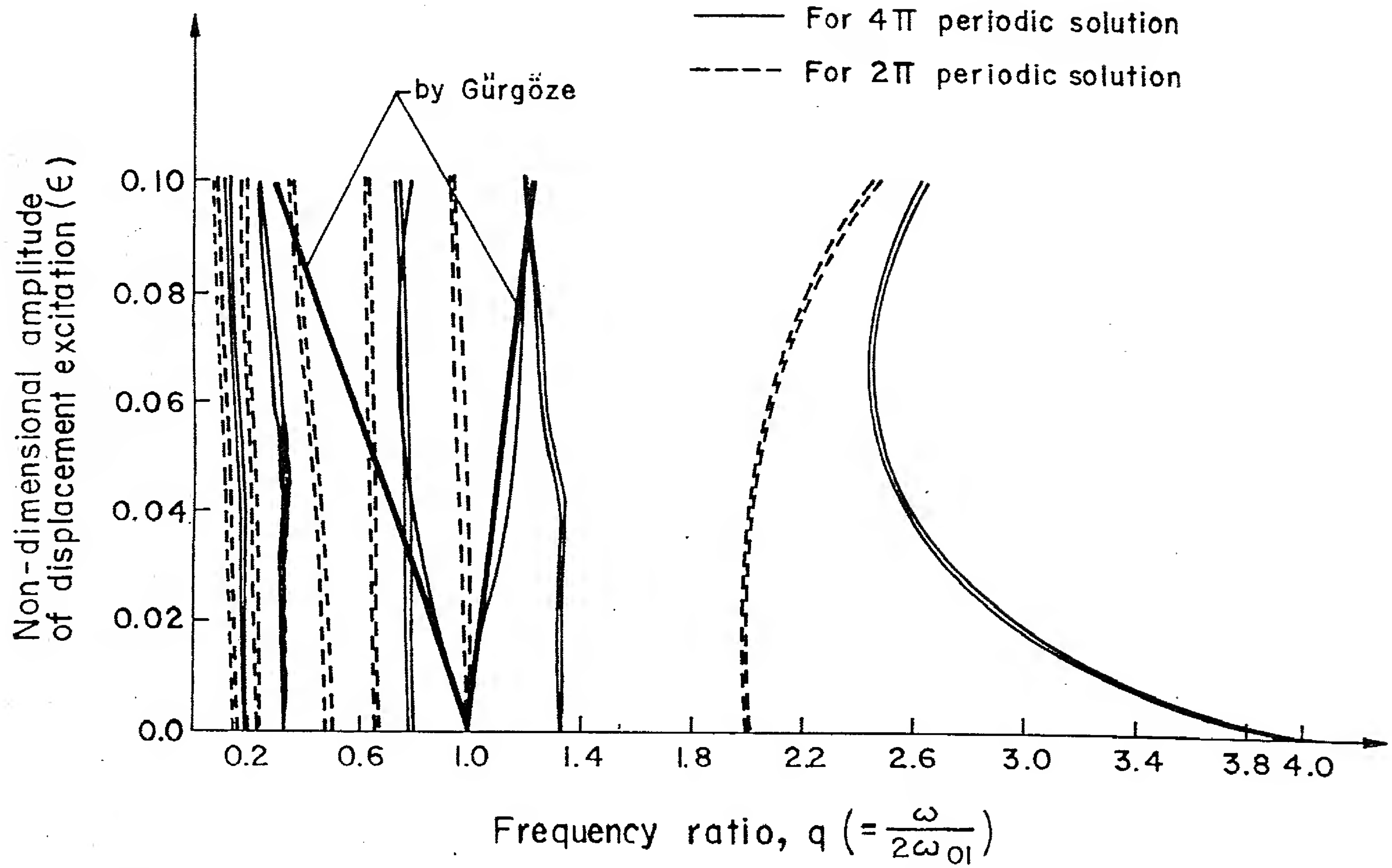

Fig. 3b Instability regions for two mode approximation (compared with PPIR by Gürgöze) for $M_* = 0.1$, $K_{f*} = 50$ and $\frac{F_o}{P_{cr}} = 0.0$ and $\delta_d = 0.0$

CHAPTER 4

METHOD OF HARMONIC BALANCE

The method of harmonic balance is to express the periodic solutions of the differential equation (D.E.) of motion by Fourier series (F.S.) , substituting this in the D.E. and equating the coefficient of each of the harmonic to zero to get a system of algebraic equations relating the frequency and the amplitudes of different harmonics of the system response. The accuracy of the resulting solution depends on the value of the amplitude of first harmonic and the number of harmonics assumed in the solution.

In this chapter, the steady state response for one mode approximation at the principal parameteric resonance are obtained, one similar to that of Gurgoze [14]. To do this the F.S. solution is put into the governing D.E. of motion for one mode approximation and the coefficients of Sine and Cosine terms in the resulting equation are equated to zeros. This results in two simultaneous homogeneous algebraic equations in two variables (coefficients of Sine and Cosine terms in the F.S.) because the governing D.E. is homogeneous and it has cubic non-linearities. These two algebraic equations are non-linear in nature but by introducing the amplitude of steady state vibration, one can make it a system

of two linear homogeneous algebraic equations. For a non-trivial solution of the transverse vibration of the beam, the determinant of the coefficient matrix must be zero. That gives a quadratic equation in the square of the amplitude of the steady state vibration as a function of system parameters M_*, K_{f*}, ϵ and frequency ratio q.

4.1 ANALYSIS:

The differential equation (2.18) , obtained from one mode approximation of the latral vibration of the beam is rewritten as

$$\ddot{\bar{B}}(t) + \frac{c}{\mu}\dot{\bar{B}}(t) + (\lambda + \gamma \text{ Cos } \omega t)\,\bar{B}(t) + \psi_1(\bar{B},\dot{\bar{B}},\ddot{\bar{B}}) = 0 \qquad \text{.... (4.1)}$$

where,

$$t = \frac{z}{\omega}$$

$$\lambda = \omega_{01}^2 - \epsilon K_{f2},$$

$$\gamma = \epsilon\,(K_{f2} - M_{22}\omega^2),$$

$$\delta = \frac{\pi^2}{8}\omega_o^2 + K_{f4},$$

$$\psi_1(\bar{B},\dot{\bar{B}}\ \ddot{\bar{B}}) = \delta\,\bar{B}^3 - \frac{3\pi^2}{8}\,\frac{c}{\mu}\,\bar{B}^2\,\dot{\bar{B}} + M_{46}\,\bar{B}\,\dot{\bar{B}}^2 + M_{44}\,\bar{B}^2\,\ddot{\bar{B}} \qquad \text{.... (4.2)}$$

and the overdots denote derivatives with respect to time t.

In the non-linear system, one obtains periodic solution with frequency equal to that of the excitation Ω as well as Ω/n, n=2,3,..., called subharmonic oscillations. When the excitation frequency happens to be equal to that

to be equal to that of the subharmonic, one experiences the phenomenon of resonances, called subharmonic resonances. Similarly, superharmonic oscillations are experienced. When the frequency of excitation is close to one half of the frequency of free oscillation term and the system response is very large, it is called the principal parametric resonance.

To investigate the steady state response in the neighbourhood of principal parametric resonance, that is $\omega \approx 2\omega_{o1}$ assume the 4π - periodic solution of equation (4.1) as

$$\bar{B}(t) = a_1 \sin\left(\frac{\omega t}{2}\right) + b_1 \cos\left(\frac{\omega t}{2}\right) \qquad \ldots. (4.3)$$

Substituting (4.3) into (4.1) and setting the coefficient of $\sin\left(\frac{\omega t}{2}\right)$ and $\cos\left(\frac{\omega t}{2}\right)$ separately equal to zero the following equations are obtained.

$$(1-3\mu_* + 4m_1 q^2)\, a_1 - 2\delta_d q b_1 + a^2\left[\left(\frac{3\delta}{4\omega_{o1}^2}\right) a_1 + q\, D_L b_1 - Kq^2 a_1\right] = 0$$

$$\ldots..(4.4)$$

$$(1-\mu_* - 4\, m_2 q^2) b_1 + 2\delta_d q a_1 + a^2\left[\left(\frac{3\delta}{4\omega_{o1}^2}\right) b_1 - q\, D_L a_1 - Kq^2 b_1\right] = 0$$

where, a_1 and b_1 comprise the amplitude and phase,

$a = \sqrt{a_1^2 + b_1^2}$ = amplitude of steady state vibration,

$$\mu_* = \frac{\epsilon K_{f2}}{2\omega_{o1}^2} = \frac{G}{2\pi^2} \qquad \frac{K_{f*}}{1-\frac{F_o}{P_{cr}}},$$

$$M_* = \frac{M}{\mu L},$$

$$m_1 = \frac{1}{2}(\varepsilon M_{22} - \frac{1}{2}) = \varepsilon(\pi^2/2)(M_* + \frac{1}{2}) - \frac{1}{4},$$

$$m_2 = \frac{1}{2}(\varepsilon M_{22} + \frac{1}{2}) = \varepsilon(\pi^2/2)(M_* + \frac{1}{2}) + \frac{1}{4},$$

$$K = \frac{\pi^2}{4}[\pi^2 M_* - \frac{\pi^2}{3} - \frac{3}{4}]$$

and $$D_L = \frac{3\pi^2}{16}\delta_d \qquad \ldots\ldots (4.5)$$

Equation (4.4) has obvious solution $a_1=b_1=a=0$. For a non-trivial solution consider equations (4.4) as a system of linear homogeneous algebraic equations in 'a_1' and 'b_1' involving 'a' and set the determinant of its coefficient matrix equal to zero. This gives a quadratic equation in a^2 having the solution

$$a = ([(m_* S_* + 2\delta_d q^2 D_L) \pm [(m_* S_* + 2\delta_d q^2 D_L)^2 - (m_{**} + 4\delta_d^2 q^2)(S_*^2 + q^2 D_L^2)]^{1/2}]/(S_*^2 + q^2 D_L^2))^{1/2} \qquad \ldots\ldots (4.6)$$

where

$$S_* = Kq^2 - \delta_*$$

$$\delta_* = \frac{3\delta}{4\omega_{o1}^2} = \frac{3\pi^2}{16}[\frac{1}{2} + \frac{K_{f*}}{\pi^2}]/(1 - \frac{F_o}{P_{cr}}),$$

$$m_* = 1 - 2\mu_* - q^2$$

and
$$m_{**} = (1-\mu_*)(1-3\mu_*) + 4q^2[m_1(1-\mu_*) - m_2(1-3\mu_*)] - 16 m_1 m_2 q^4 \qquad \ldots\ldots (4.7)$$

The equation (4.6) is an explicit equation for 'a' as a function of q, the freqency ratio. It is called the response curve.

The backbone curves, which indicate the relation between the bending eigen frequency ω and the amplitude 'a' of the non-linear beam system are obtained, equating terms to zero which correspond to damping and displacement excitation. To do this, replace $\frac{\omega}{2}$ by ω in equations (4.1) and (4.3) and proceed in the same way as that for dynamic response. This is equivalent to setting $\delta_d = D_L = \varepsilon = 0$ in the equation (4.6) and getting

$$a = [(1-q^2)/(Kq^2 - \delta_*)]^{1/2} \qquad \ldots\ldots \ (4.8)$$

where $q = \frac{\omega}{\omega_{o1}}$ = ratio of eigen frequencies of the non-linear and linear systems respectively.

4.2 RESULTS:

Backbone curves obtained from equation (4.8) are graphically represented in Figs. 4(a) and 4(b) for

(i) $\varepsilon = 0$, $M_* = 0.1$, $\frac{F_o}{P_{cr}} = 0.5$, $\delta_d = 0$ and K_{f*} as a parameter ranging from 0 to 50 and

(ii) $\varepsilon = 0$, $K_{f*} = 15$, $\frac{F_o}{P_{cr}} = 0.5$, $\delta_d = 0$ and M_* as a parameter ranging from 0 to 0.4, respectively.

From equation(4.6) the response curves are plotted for the following cases

(i) $\epsilon = 0.05$, $M_* = 0.1$, $\delta_d = 0$, $\frac{F_o}{P_{cr}} = 0.5$ and stiffness ratio as a parameter ranging from 0 to 50 , See Fig. 4 (c)

(ii) $\epsilon = 0.05$, $K_{f*} = 2$, $\delta_d = 0$, $\frac{F_o}{P_{cr}} = 0.5$ and mass ratio M_* as a parameter ranging from 0 to 0.4, See Fig. 4(d).

(iii) $M_* = 0.1$, $K_{f*} = 15$, $\delta_d = 0$, $\frac{F_o}{P_{cr}} = 0.5$ and excitation amplitude ϵ as a parameter ranging from 0.0 to 0.05, See Fig. 4(e) .

Here both the stable and unstable branches of response curves are found out. If (+) ve sign is taken in place of ($\pm$) sign in the equation (4.6) stable branches are obtained and for unstable branches (−)ve sign is taken. The stable and unstable branches are shown by firm and dotted curves respectively.

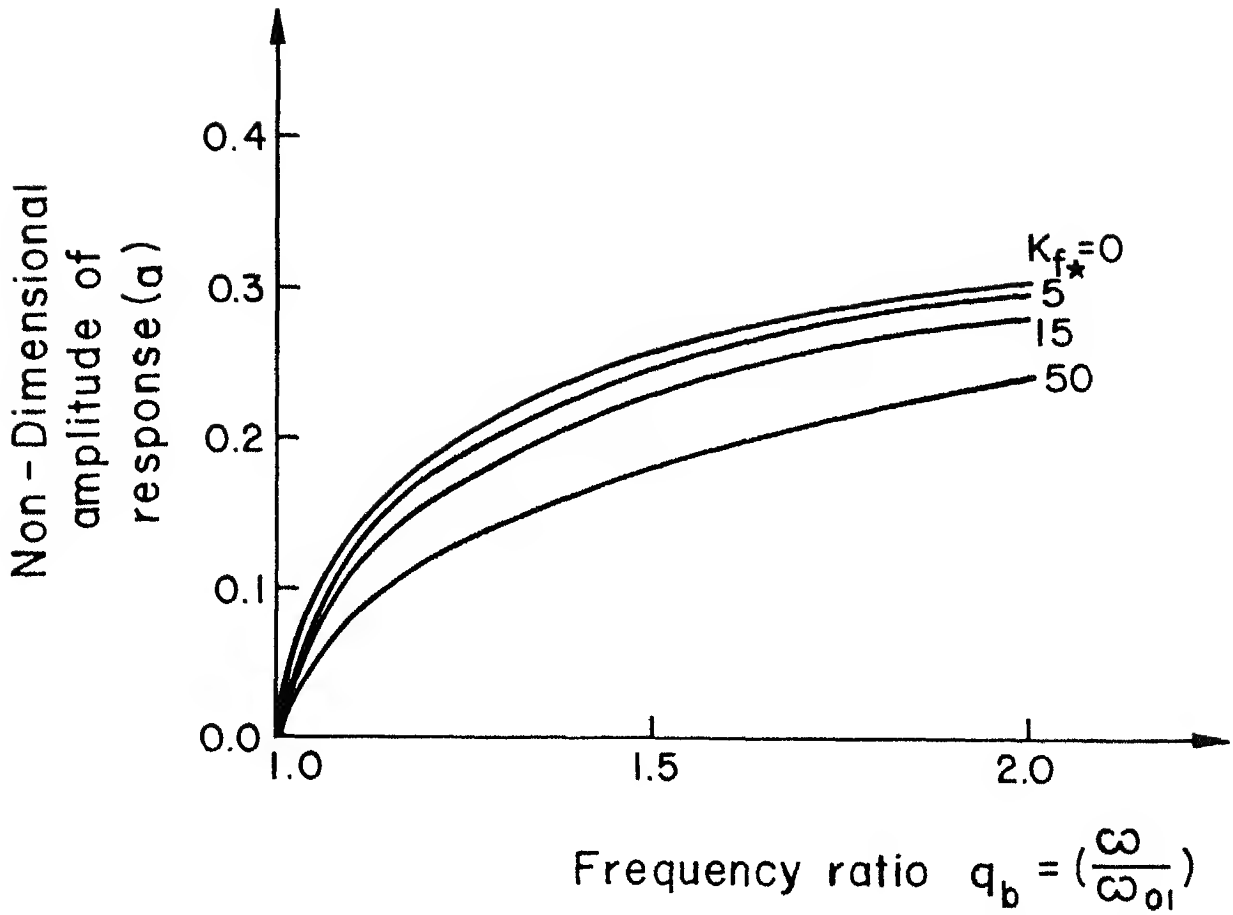

Fig. 4 a. Backbone curves for various stiffness ratios with $M_* = 0.1$, $\epsilon = 0$, $\frac{F_0}{P_{cr}} = 0.5$ and $\delta_d = 0$

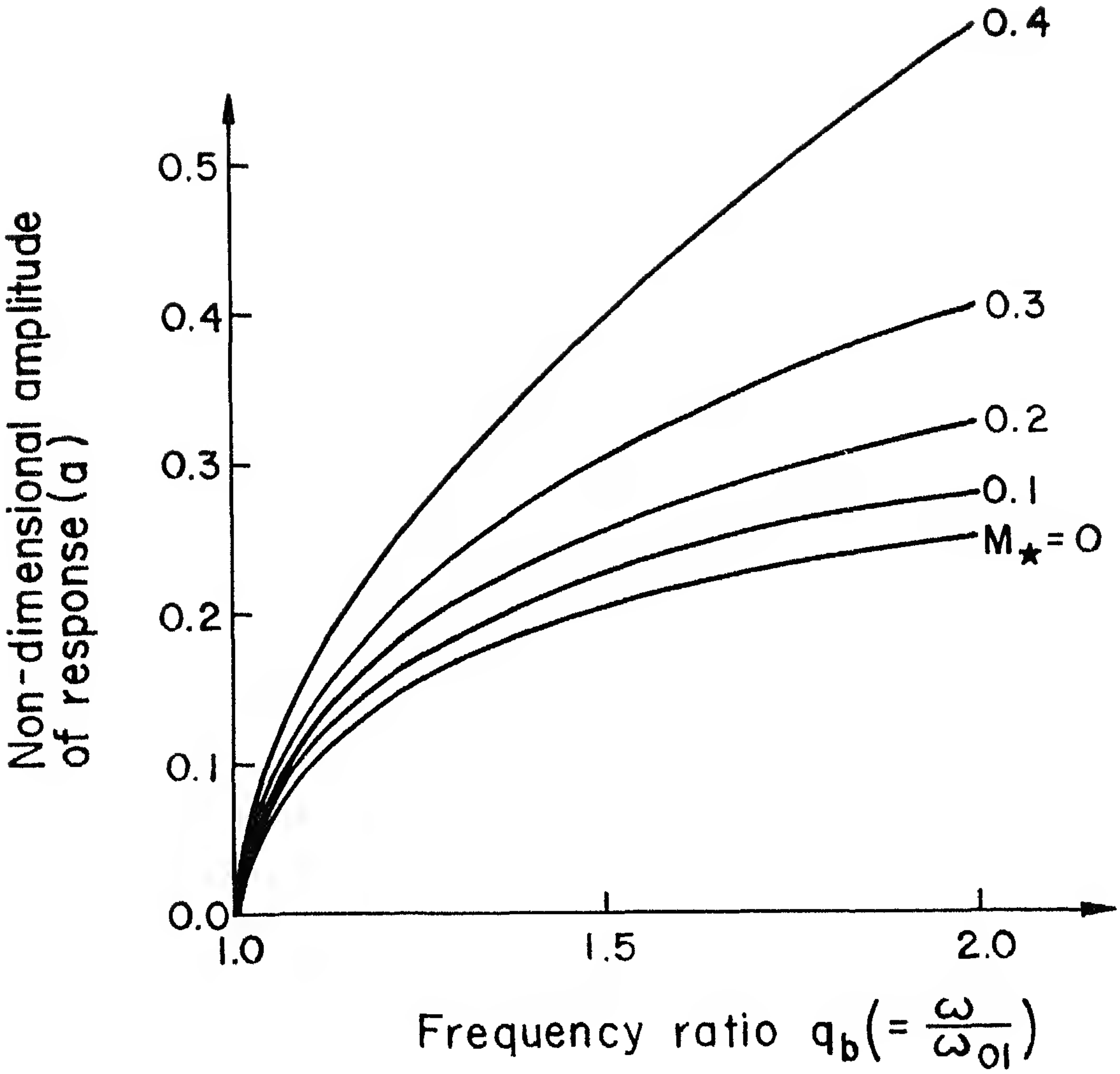

Fig. 4b Backbone curves for various mass ratios with $\epsilon = 0$, $K_{f\star} = 15$, $\frac{F_o}{P_{cr}} = 0.5$, $\delta_d = 0$

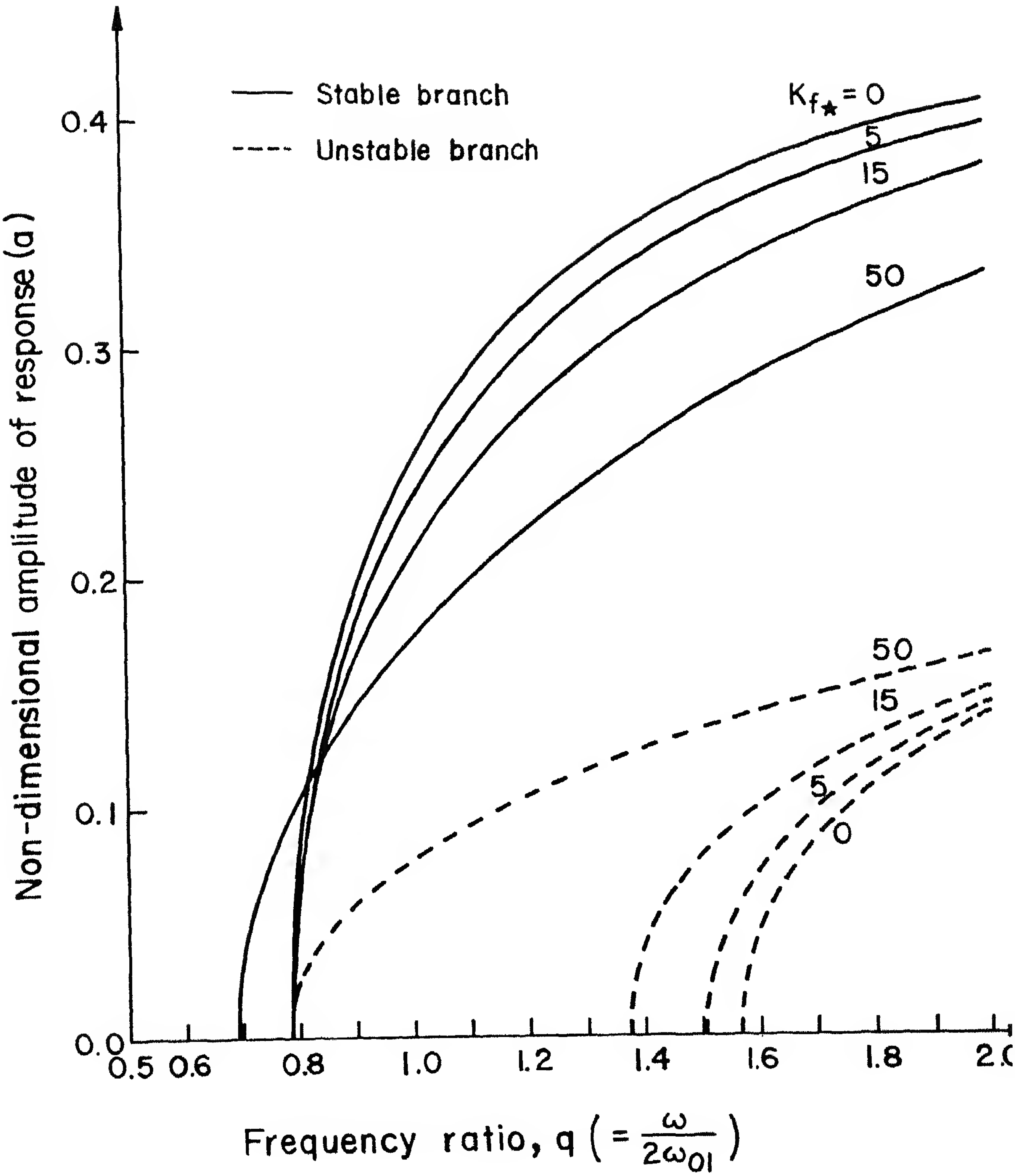

Fig. 4c Response curves for various stiffness ratios and $\epsilon = 0.05$, $M_\star = 0.1$, $\frac{F_0}{P_{cr}} = 0.5$, $\delta_d = 0.0$

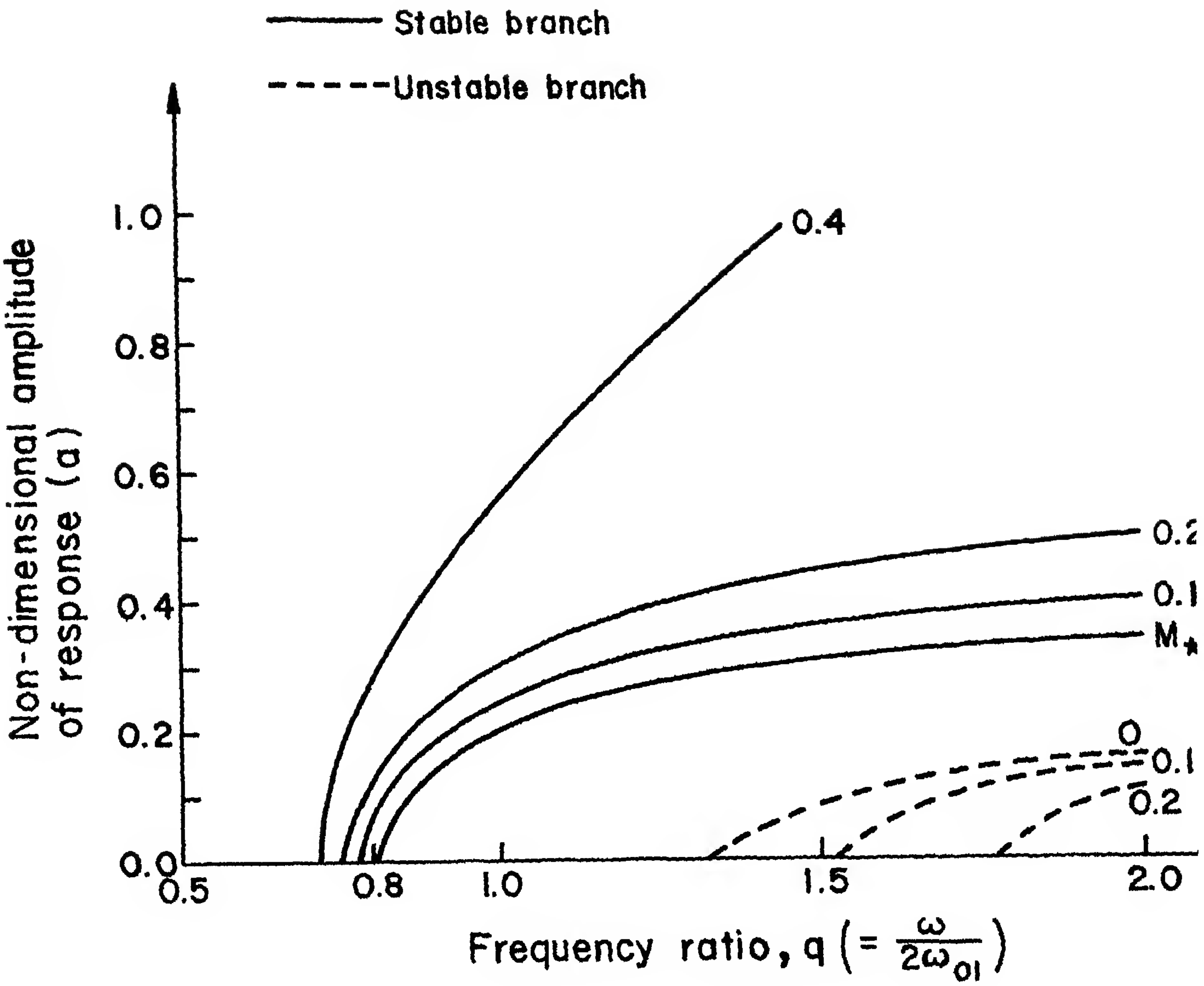

Fig. 4d Response curves for various mass ratios and $\epsilon = 0.05$, $K_{f*} = 2$, $\frac{F_0}{P_{cr}} = 0.5$, $\delta_d = 0$

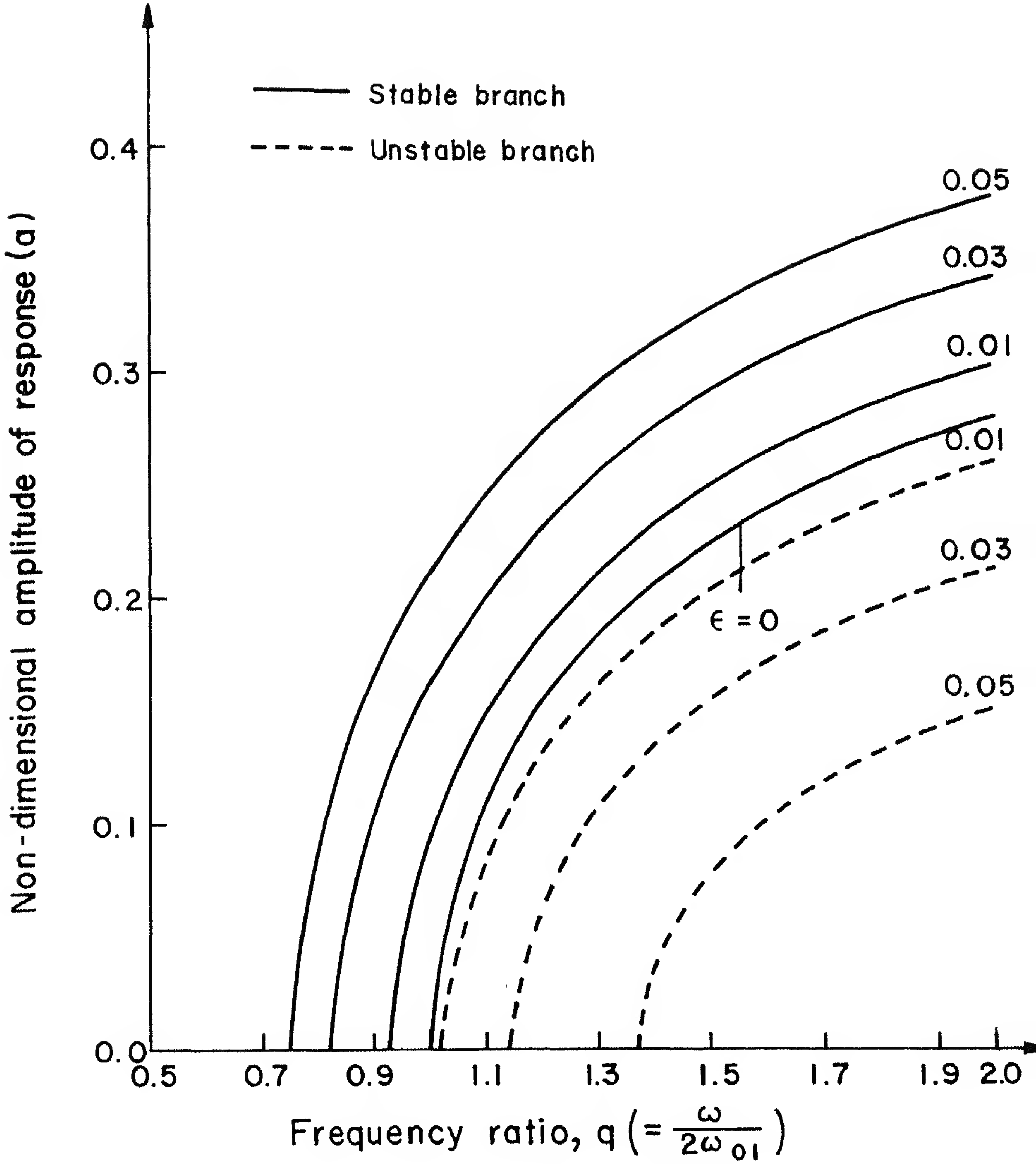

Fig. 4e Response curves for various excitation amplitudes and $M_\star = 0.1$, $K_{f\star} = 15$, $\frac{F_o}{P_{cr}} = 0.5$ $\delta_d = 0$

CHAPTER 5

ANALYSIS BY METHOD OF SLOWLY VARYING PARAMETERS

In this chapter, the steady state response characteristics in the neighbourhood of main parameteric instability i.e. $\omega \simeq 2\omega_{o1}$ is found by the method of slowly varying parameters. The backbone curves are found out for $\omega \simeq \omega_{o1}$.

For linear undamped free systems the solution can be written as a Sine or Cosine function of time with constant amplitude, phase angle and circular frequency. However, for weakly non-linear system with small damping the solution for the displacement and velocity can be expressed like that of the linear case but with a slowly varying amplitude and phase. 'Slowly' means that amplitude and phase remain nearly constant during the period of oscillation. This can be viewed as a transformation from one dependent variable to two dependent variables namely the amplitude and phase. So from this transformation, one gets two non-autonomous differential equations in two dependent variables, amplitude and phase. These non-autonomous differential equations are transformed to autonomous system by taking average values of the right hand side over the time period. It is presumed that the solution of these two autonomous equations represent approximately

the solution of the non-autonomous equations thus leading to the steady state response of the system. The backbone curve is found out in a similar way.

5.1 ANALYSIS :

For one mode approximation of lateral vibration of the beam the following second order non-linear homogeneous DE was derived in Chapter 2 (equation 2.21)

$$\bar{B}_{,zz} + \frac{c}{\mu\omega}\bar{B}_{,z} + \left[\frac{\omega_{o1}^2 - \varepsilon K_{f2}}{\omega^2} + \varepsilon\left(\frac{K_{f2}}{\omega^2} - M_{22}\right)\cos z\right]\bar{B}$$
$$+ \frac{1}{\omega^2}\left(\frac{\pi^2\omega_o^2}{8} + K_{f4}\right)\bar{B}^3 + M_{46}\bar{B}\bar{B}_{;z}^2$$
$$- \frac{3\pi^2}{8} \cdot \frac{c}{\mu\omega}\bar{B}^2\bar{B}_{,z} + M_{44}\bar{B}^2\bar{B}_{,zz} = 0 \qquad \ldots\ldots(5.1)$$

For linearised version of equation (5.1) with no damping force and parametric excitation, the solution, in the neighbourhood of main parametric instability region ($\omega \simeq 2\omega_{o1}$), can be written as

$$\bar{B} = a \sin\left(\frac{z}{2} + \psi\right)$$

Where 'a' and 'ψ' are constants. But for using the method of slowly varying parameters the solution of equation (5.1) should be written as

$$\bar{B} = a(z) \sin\left[\frac{z}{2} + \psi(z)\right] \qquad \ldots\ldots(5.2)$$

where amplitude 'a' and phase 'ψ' are time dependent.

As stated earlier it is assumed that the velocity

$$\bar{B}' = \frac{1}{2} a(z) \cos\left[\frac{z}{2} + \psi(z)\right] \qquad \ldots.(5.3)$$

where subscript dash means differentiation with respect to new time variable z.

Differentiating (5.2) with respect to z and comparing it with (5.3) one gets

$$a' \sin Q + a\,\psi' \cos Q = 0 \qquad \ldots.(5.4)$$

where, $Q = \frac{z}{2} + \psi(z)$.

Differentiating (5.3) with respect to z

$$\bar{B}'' = \frac{a'}{2} \cos\left(\frac{z}{2}+\psi\right) - \frac{a}{2}\left(\frac{1}{2}+\psi'\right) \sin\left(\frac{z}{2}+\psi\right) \qquad \ldots.(5.5)$$

Substituting (5.2), (5.3) and (5.5) in equation (5.1) one gets.

$$-\frac{1}{2}\left(a'\cos Q - a\psi' \sin Q\right) = -\frac{a}{4}\sin Q + \frac{c}{2\mu\omega}\, a \frac{\cos Q}{1+M_{44}a^2\sin^2 Q}$$

$$+ a\left[\frac{\omega_{o1}^2 - \varepsilon K_{f2}}{\omega^2} + \varepsilon\left(\frac{K_{f2}}{\omega^2} - M_{22}\right)\cos z\right]\frac{\sin Q}{1+M_{44}a^2\sin^2 Q}$$

$$+ \frac{1}{\omega^2}\left(\frac{\pi^2\omega_o^2}{8} + K_{f4}\right) a^3 \frac{\sin^3 Q}{1+M_{44}a^2\sin^2 Q}$$

$$+ \frac{M_{46}}{4} a^3 \frac{\sin Q\cos^2 Q}{1+M_{44}a^2\sin^2 Q} - \frac{3\pi^2}{16}\frac{c}{\mu\omega} a^3 \frac{\sin^2 Q\cos Q}{1+M_{44}a^2\sin^2 Q}$$

$$\ldots\ldots(5.6)$$

Solving a' and ψ' from equations (5.4) and (5.6)

$$-a' = -\frac{1}{2} a \operatorname{Sin} Q \operatorname{Cos} Q + \frac{c}{\mu\omega} a \frac{\operatorname{Cos}^2 Q}{1+M_{44}a^2\operatorname{Sin}^2 Q}$$

$$+ 2a \left[\frac{\omega_{o1}^2 - \varepsilon K_{f2}}{\omega^2} + \varepsilon \left(\frac{K_{f2}}{\omega^2} - M_{22}\right) \operatorname{Cos} z\right] \frac{\operatorname{Sin} Q \operatorname{Cos} Q}{1+M_{44}a^2\operatorname{Sin}^2 Q}$$

$$+ \frac{2}{\omega^2} \left(\frac{\pi^2\omega_o^2}{8} + K_{f4}\right) a^3 \frac{\operatorname{Sin}^3 Q \operatorname{Cos} Q}{1+M_{44}a^2\operatorname{Sin}^2 Q}$$

$$+ \frac{M_{46}}{2} a^3 \frac{\operatorname{Sin} Q \operatorname{Cos}^3 Q}{1+M_{44}a^2\operatorname{Sin}^2 Q} - \frac{3\pi^2}{8} \frac{c}{\mu\omega} a^3 \frac{\operatorname{Sin}^2 Q \operatorname{Cos}^2 Q}{1+M_{44}a^2\operatorname{Sin}^2 Q} \quad \ldots(5.7)$$

and

$$\psi' = -\frac{1}{2} \operatorname{Sin}^2 Q + \frac{c}{\mu\omega} \frac{\operatorname{Sin} Q \operatorname{Cos} Q}{1+M_{44}a^2 \operatorname{Sin}^2 Q}$$

$$+ 2 \left[\frac{\omega_{o1}^2 - \varepsilon K_{f2}}{\omega^2} + \varepsilon \left(\frac{K_{f2}}{\omega^2} - M_{22}\right) \operatorname{Cos} z \right] \frac{\operatorname{Sin}^2 Q}{1+M_{44}a^2\operatorname{Sin}^2 Q}$$

$$+ \frac{2}{\omega^2} \left(\frac{\pi^2\omega_o^2}{8} + K_{f4}\right) a^2 \frac{\operatorname{Sin}^4 Q}{1+M_{44}a^2\operatorname{Sin}^2 Q}$$

$$+ \frac{M_{46}}{2} a^2 \frac{\operatorname{Sin}^2 Q \operatorname{Cos}^2 Q}{1+M_{44}a^2 \operatorname{Sin}^2 Q} - \frac{3\pi^2}{8} \frac{c}{\mu\omega} a^2 \frac{\operatorname{Sin}^3 Q \operatorname{Cos} Q}{1+M_{44}a^2 \operatorname{Sin}^2 Q}$$

$$\ldots\ldots(5.8)$$

Upto this point, no more than a co-ordinate transformation has been carried out and the differential equations (5.7) and (5.8) are still exactly equivalent to (5.1). If the non-linear terms are small then a' and ψ' are also small, that is, the amplitude 'a' and phase 'ψ' change only slowly. 'Slowly' here means that the value of ψ in the argument $(\frac{z}{2} + \psi)$ as well

as the value of $a(z)$ remain nearly constant during a time interval of duration $T_o = 2\pi/(\frac{1}{2}) = 4\pi$. Since it is generally not possible to obtain an exact solution to (5.7) and (5.8) these equations are simplified by replacing the right hand sides by their temporal mean over the interval $[t, t+T_o]$. In forming the mean value, a and ψ are kept constant on the right hand side of (5.7) and (5.8). So equations (5.7) and (5.8) are replaced by

$$-a' = \frac{1}{4\pi}\Big[-\frac{a}{4}\int_0^{4\pi} \sin 2(\tfrac{z}{2}+\psi)\,dz$$

$$+ \frac{c}{\mu\omega a}\int_0^{4\pi} \frac{\cos^2(\frac{z}{2}+\psi)}{1+M_{44}a^2\sin^2(\frac{z}{2}+\psi)}\,dz$$

$$+ a\int_0^{4\pi}\Big[\frac{\omega_{o1}^2 - \varepsilon K_{f2}}{\omega^2} + \varepsilon\Big(\frac{K_{f2}}{\omega^2} - M_{22}\Big)\cos z\Big]\frac{\sin 2(\frac{z}{2}+\psi)}{1+M_{44}a^2\sin^2(\frac{z}{2}+\psi)}\,dz$$

$$+ \frac{2}{\omega^2}\Big(\frac{\pi^2\omega_o}{8} + K_{f4}\Big)a^3\int_0^{4\pi} \frac{\sin^3(\frac{z}{2}+\psi)\cos(\frac{z}{2}+\psi)}{1+M_{44}a^2\sin^2(\frac{z}{2}+\psi)}\,dz$$

$$+ \frac{M_{44}}{2}a^3\int_0^{4\pi} \frac{\sin(\frac{z}{2}+\psi)\cos^2(\frac{z}{2}+\psi)}{1+M_{44}a^2\sin^2(\frac{z}{2}+\psi)}\,dz$$

$$-\frac{3\pi^2}{8}\frac{c}{\mu\omega}a^3\int_0^{4\pi} \frac{\sin^2(\frac{z}{2}+\psi)\cos^2(\frac{z}{2}+\psi)}{1+M_{44}a^2\sin^2(\frac{z}{2}+\psi)}\,dz\Big] \qquad \ldots\ldots(5.9)$$

and

$$\psi' = \frac{1}{4\pi}[-\frac{1}{2} \int_0^{4\pi} \sin^2(\frac{z}{2}+\psi)dz + \frac{c}{\mu\omega} \int_0^{4\pi} \frac{\sin(\frac{z}{2}+\psi)\cos(\frac{z}{2}+\psi)}{1+M_{44}a^2\sin^2(\frac{z}{2}+\psi)} dz$$

$$+2\int_0^{4\pi}[\frac{\omega_{o1}^2-\epsilon K_{f2}}{\omega^2} + \epsilon(\frac{K_{f2}}{\omega^2} - M_{22})\cos z] \frac{\sin^2(\frac{z}{2}+\psi)}{1+M_{44}a^2\sin^2(\frac{z}{2}+\psi)} dz$$

$$+\frac{2a^2}{\omega^2}(\frac{\pi^2\omega_{o1}^2}{8} + K_{f4})\int_0^{4\pi}\frac{\sin^4(\frac{z}{2}+\psi)}{1+M_{44}a^2\sin^2(\frac{z}{2}+\psi)} dz$$

$$+\frac{M_{46}}{2}a^2\int_0^{4\pi}\frac{\sin^2(\frac{z}{2}+\psi)\cos^2(\frac{z}{2}+\psi)}{1+M_{44}a^2\sin^2(\frac{z}{2}+\psi)} dz$$

$$-\frac{3\pi^2}{8}\frac{c}{\mu\omega}a^2\int_0^{4\pi}\frac{\sin^3(\frac{z}{2}+\psi)\cos(\frac{z}{2}+\psi)}{1+M_{44}a^2\sin^2(\frac{z}{2}+\psi)} dz] \quad \ldots\ldots 5.10)$$

respectively.

All the definite integrals in equations (5.9) and (5.10) are calculated subject to the condition

$$1+M_{44}a^2 > 0$$

where M_{44} is negative for mass ratio less than 0.37 and it is the coefficient of non-linear inertia of the concentrated mass. Therefore for a high value of 'a' the method becomes invalid. But the results can be obtained satisfactorily for a definite range of amplitude. For steady state solutions of the autonomous equations (5.9) and (5.10)

$$\psi' = a' = 0$$

giving

$$(M_{44}\ a^2+2-2\sqrt{M_{44}\ a^2+1})\ (\mu_* - 2Eq^2M_{22})\operatorname{Sin} 2\psi$$

$$= M_{44}\ a^2[1-\sqrt{M_{44}\ a^2+1} + \frac{3\pi^2}{16M_{44}}(M_{44}a^2+2-2\sqrt{M_{44}\ a^2+1})]q\delta_d \qquad \text{....(5.11)}$$

and

$$(2\sqrt{M_{44}\ a^2+1} - M_{44}\ a^2-2)(\mu_*-2E\ q^2M_{22})\ \operatorname{Cos}2\psi$$

$$= \frac{M_{44}^2\ a^4}{4}\sqrt{M_{44}\ a^2+1}\ \ [q^2- \frac{2}{M_{44}a^2}\ (1- \frac{1}{\sqrt{M_{44}\ a^2+1}})\ (1-2\mu_*)$$

$$- \frac{1}{M_{44}}[\ 1- \frac{2}{M_{44}a^2}\ (\ 1+ \frac{1}{\sqrt{M_{44}\ a^2+1}}\ )]\ \frac{\delta}{\omega_{o1}{}^2}$$

$$- \frac{M_{46}}{M_{44}}[\ 1+ \frac{2(1-\sqrt{M_{44}\ a^2+1}\ )}{M_{44}\ a^2}\ ]\ \ q^2] \qquad \text{.....(5.12)}$$

where

$$\delta_d = \frac{c}{2\mu\omega_{o1}}\ ,$$

$$\mu_* = \frac{EK_{f2}}{2\omega_{o1}^2}\ ,$$

$$\delta = \frac{\pi^2\omega_o^2}{8} + K_{f4}$$

and $$q = \frac{\omega}{2\omega_{o1}}\ .$$

Introduce

$$A1 = M_{44}\ a^2+2-2\sqrt{M_{44}\ a^2+1},$$

$$B1 = \sqrt{M_{44}\ a^2+1}\ ,$$

$$C1 = (\mu_* - 2\varepsilon q^2 M_{22}).A1,$$

$$D1 = M_{44}\ a^2[1-B1 + \frac{A_1}{M_{44}} \cdot \frac{3\pi^2}{16}]\ q\delta_d$$

and

$$D2 = \frac{M_{44}^2 a^4}{4} B1.\ [q^2- \frac{2}{M_{44}a^2}(1-\frac{1}{B1})(1-2\mu_*)$$

$$- \frac{1}{M_{44}}[1- \frac{2}{M_{44}a^2}(1-\frac{1}{B1})]\ \frac{4\delta_*}{3} - \frac{M_{46}}{M_{44}^2}\ \frac{A1}{a^2}\ q^2]\ .$$

Therefore, the equations (5.11) and (5.12) can be written as

$$C1 \ .\ \ \mathrm{Sin}\ 2\psi = D1 \qquad .....(5.13)$$

and

$$- C1\ .\ \ \mathrm{Cos}\ 2\psi = D2 \qquad .....(5.14)$$

Equations (5.13) and (5.14) are equivalent to

$$C1^2 = D1^2 + D2^2 \qquad .....\ (5.15)$$

from which the steady state response characteristics can be obtained.

The backbone curves are obtained by substituting $\frac{\omega}{2}$ by ω in equation (5.1), assuming $\overline{B}=a(z)\ \mathrm{Sin}\ [z+\psi\ (z)]$ and proceding in the same way. Finally the equations obtained are exactly similar to (5.11) and (5.12) but here $q = \frac{\omega}{\omega_{o1}}$.

5.2 RESULTS :

Equation (5.15) is an implicit equation of amplitude 'a' and frequency ratio 'q' and it is very difficult to express 'a' as a function of 'q'. So first find out the value of q for a=0 and then find its value in the neighbourhood of a=0. The procedure for computing q for different values of 'a' in the neighbourhood of a=0 is the same as one adopted earlier for finding the instability regions, section 3.1(a).

Based on the computed results for $\varepsilon=0$, $\delta_d=0$ and $\frac{F_o}{P_{cr}}=0$, the backbone curves have been plotted for the following cases:

(i) Mass ratio $M_*=0.1$ and stiffness ratio K_{f*} from 0.0 to 50.0, See Fig 5(a).

(ii) Stiffness ratio $K_{f*}=15.0$ and mass ratio M_* from 0.0 to 0.4, see Fig 5(b).

The response curves for $\delta_d=0$ and $\frac{F_o}{P_{cr}}=0.5$ and other parameters mentioned below have been shown:

(i) Excitation amplitude $\varepsilon=0.05$, mass ratio $M_*=0.1$ and stiffness ratio K_{f*} from 0.0 to 50.0, see Fig. 5(c).

(ii) Excitation amplitude $\varepsilon=0.05$, stiffness ratio $K_{f*}=2.0$ and mass ratio M_* from 0.0 to 0.4, see Fig. 5(d).

(iii) Mass ratio $M_*=0.1$, stiffness ratio $K_{f*}=15.0$ and excitation amplitude ε from 0.0 to 0.05, see Fig. 5(e).

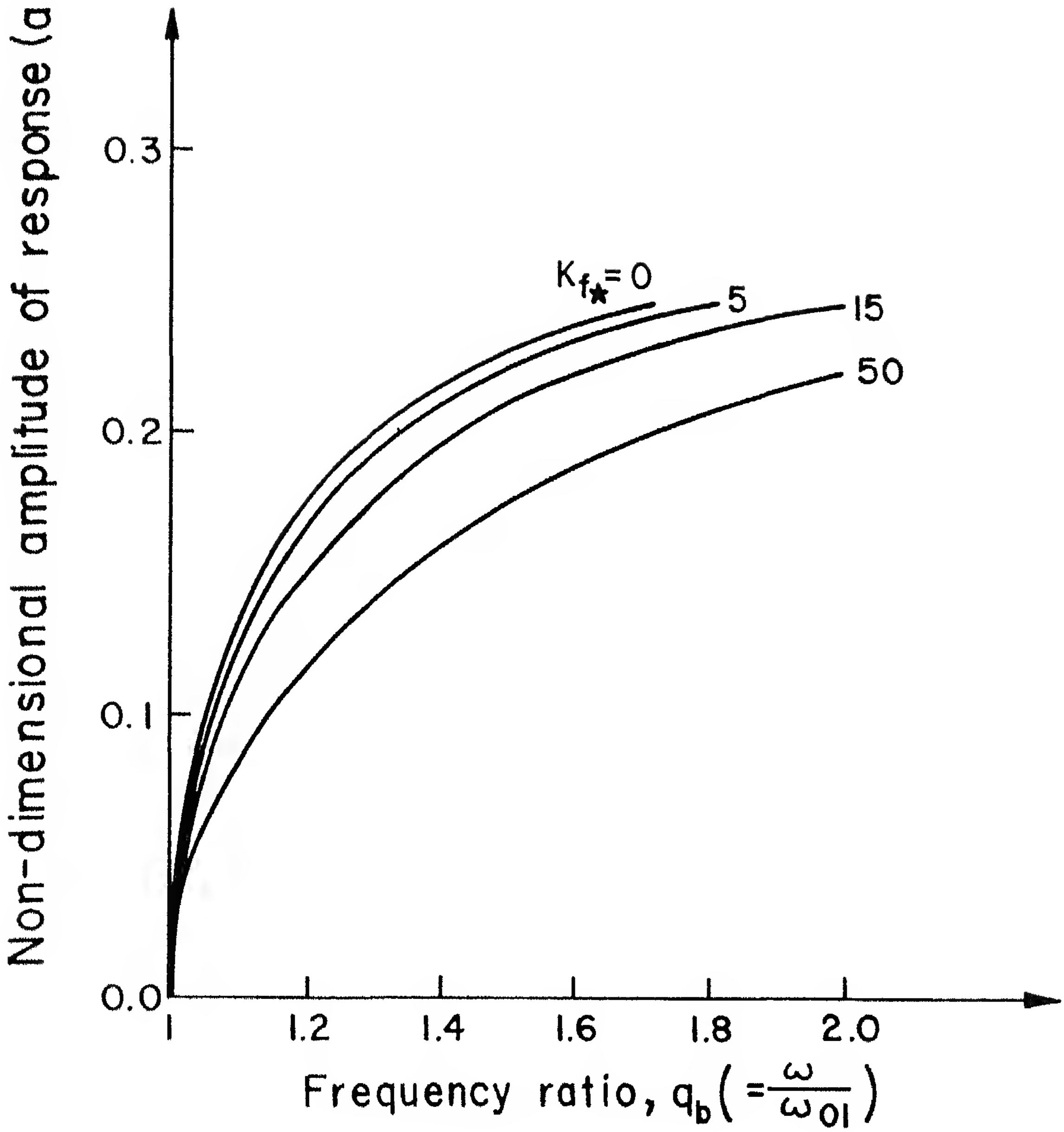

Fig. 5a Backbone curves for various stiffness ratios with $M_* = 0.1$, $\epsilon = 0$, $\frac{F_o}{P_{cr}} = 0.5$ and $\delta_d = 0$

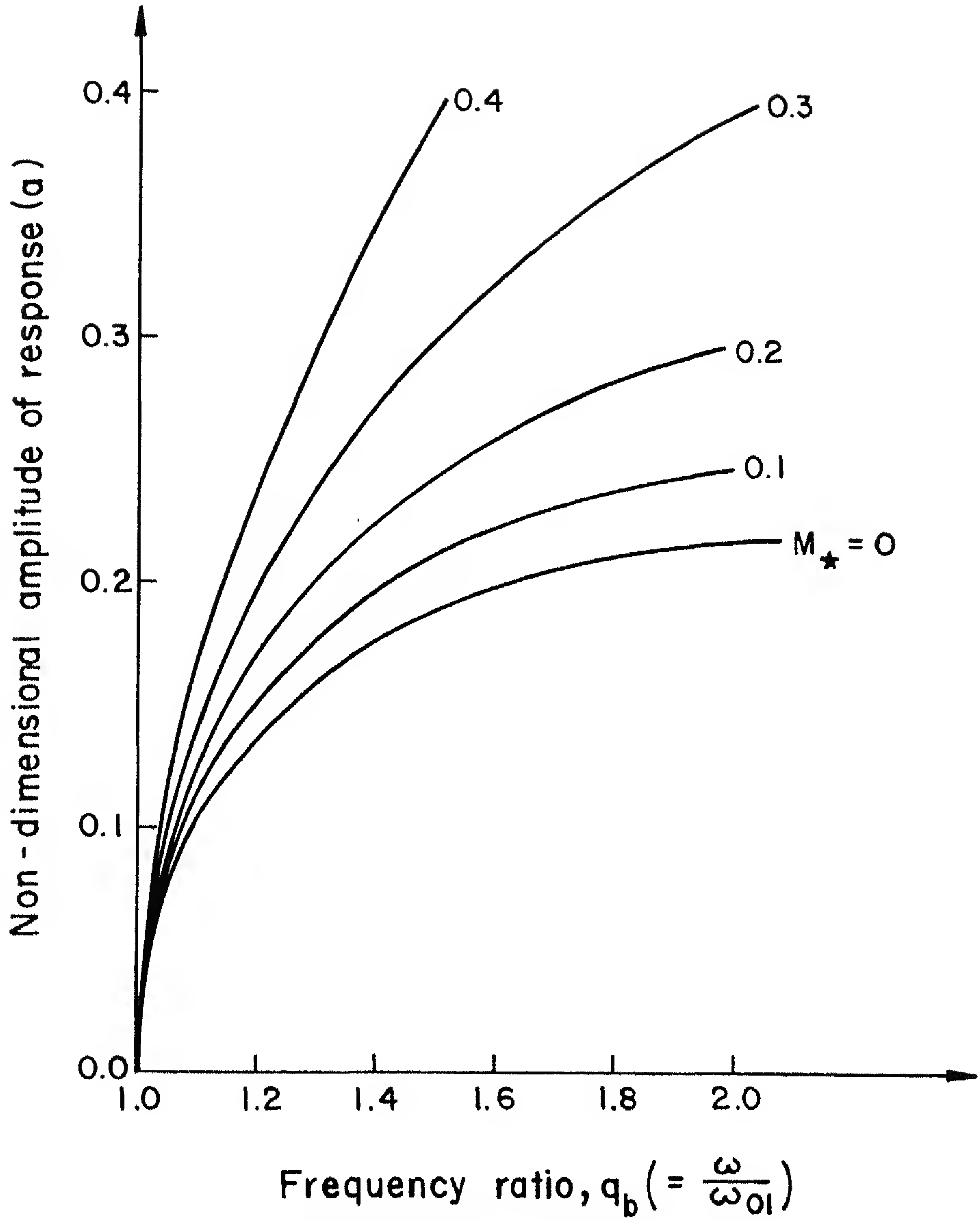

Fig. 5b Backbone curves for various mass ratios with $\epsilon = 0$, $K_{f\star} = 15$, $\frac{F_o}{P_{cr}} = 0.5$ and $\delta_d = 0.0$

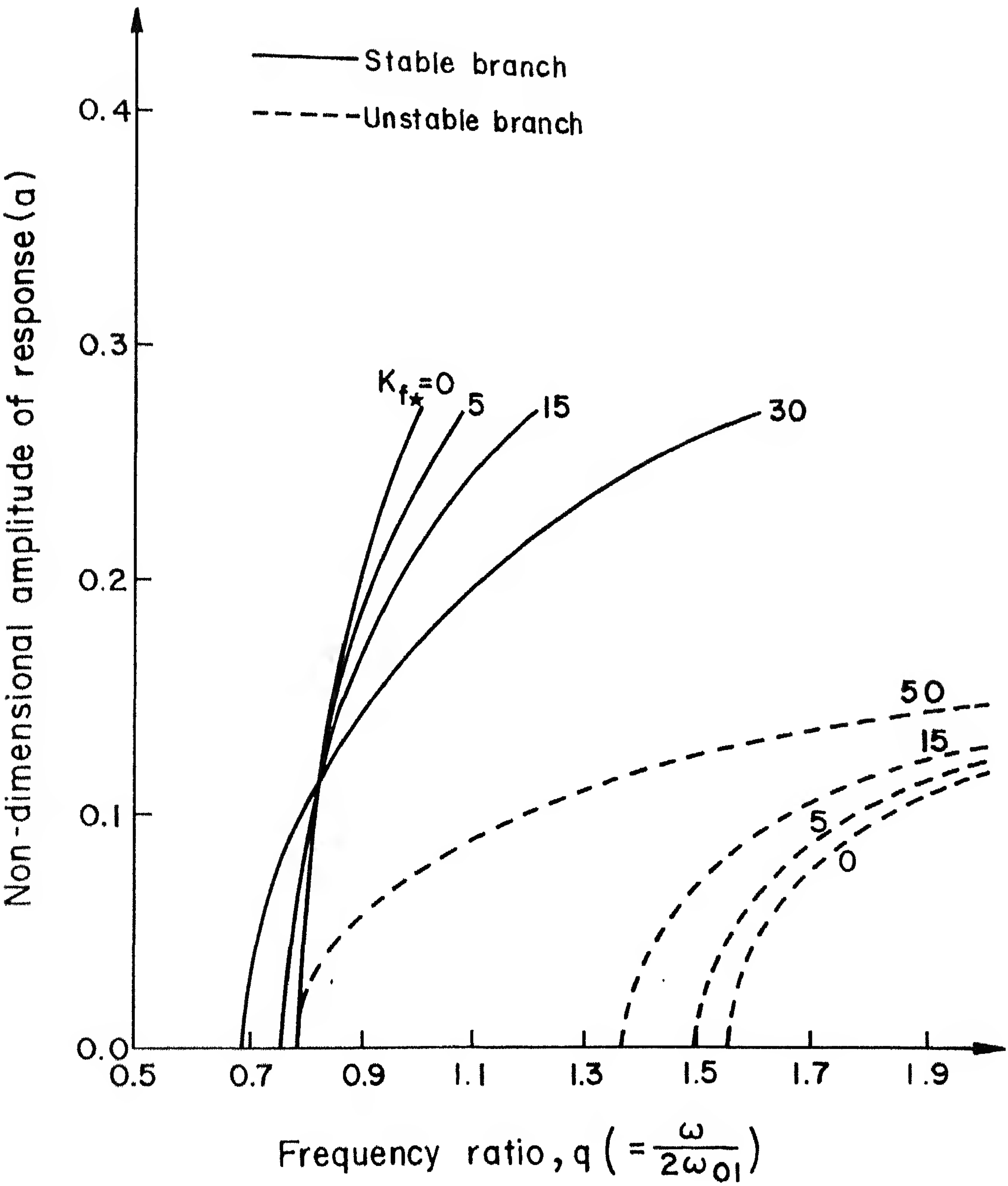

Fig. 5c Response curves for various stiffness ratios with $\epsilon = 0.05$, $M_\star = 0.1$, $\frac{F_0}{P_{cr}} = 0.5$ and $\delta_d = 0$

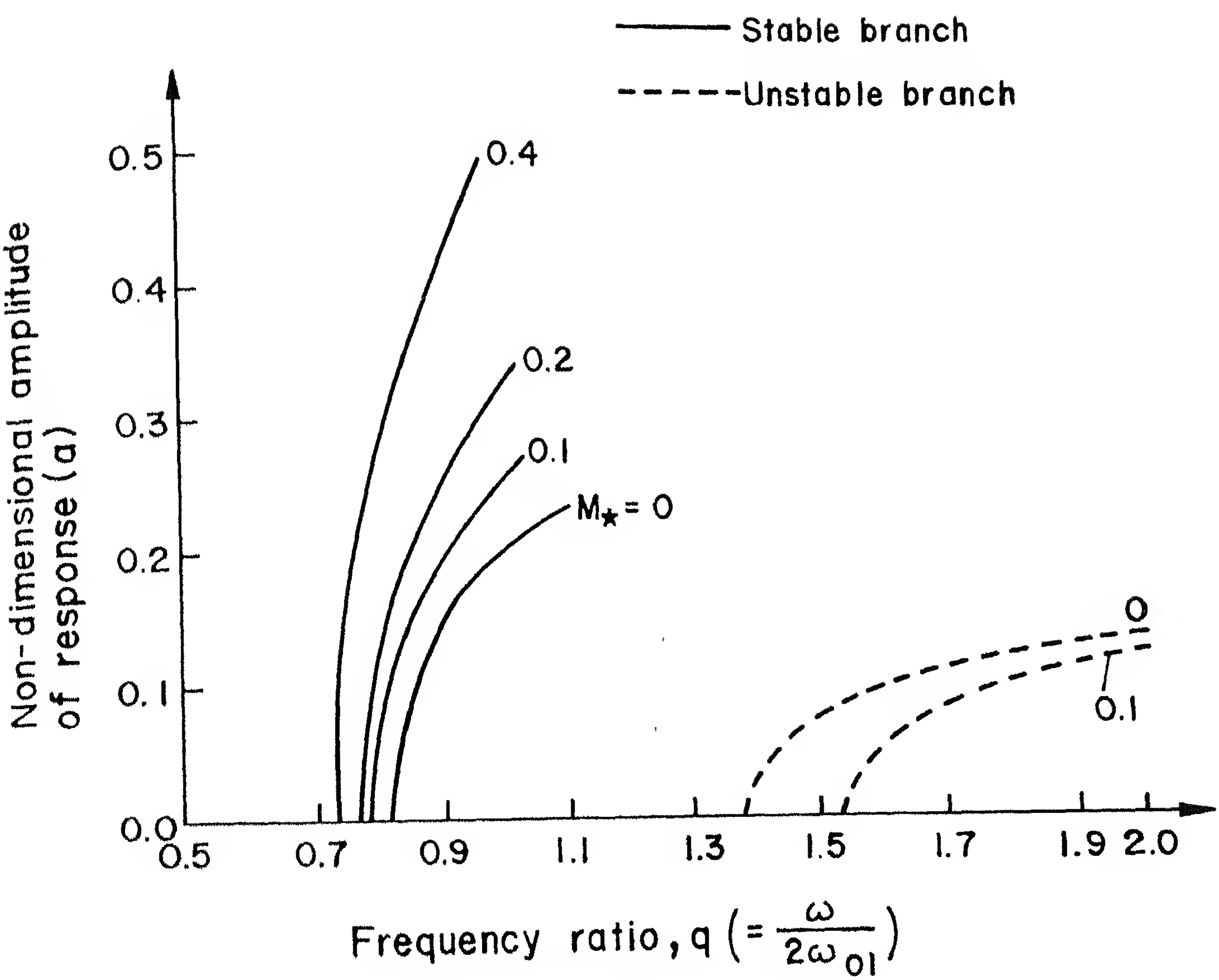

Fig. 5d Response curves for various mass ratios with $\epsilon = 0.05$, $K_{f\star} = 2$, $\frac{F_0}{P_{cr}} = 0.5$, $\delta_d = 0.0$

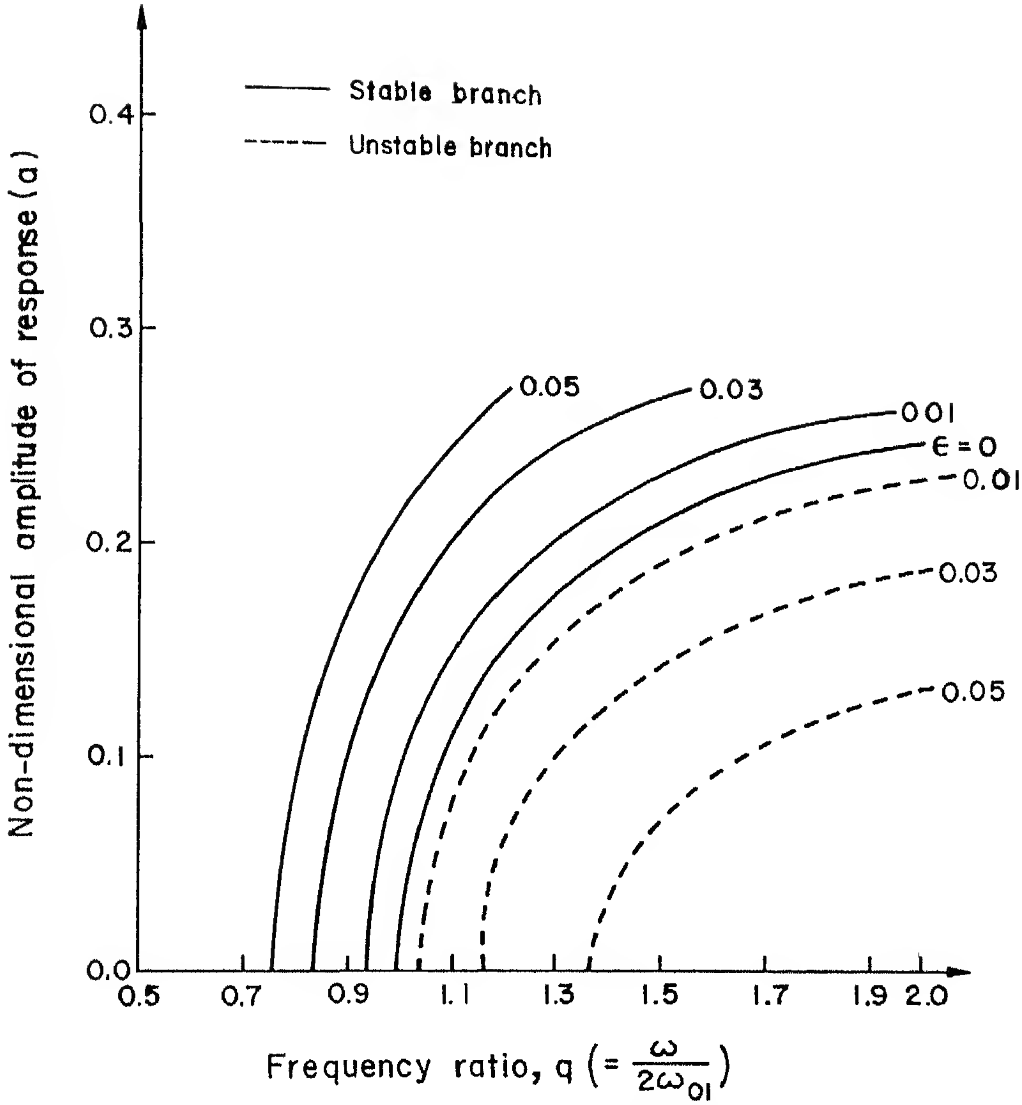

Fig. 5e Response curves for various excitation amplitudes with $M_{\star} = 0.1$, $K_{f\star} = 15$, $\frac{F_0}{P_{cr}} = 0.5$ and $\delta_d = 0.0$

CHAPTER 6

ANALYSIS BY METHOD OF MULTIPLE SCALES

In this chapter, the method of multiple scales has been adopted to obtain the steady state response of the one mode approximation of the system.

The method of multiple scales is to consider the expansion representing the response to be a function of multiple independent variables or scales instead of a single variable. The number of independent time scales needed depends on the order to which the expansion is carried out. Here, the expansion of the dependent variable $\bar{B}$ (response) is carried out only upto first order. The equation of motion for one mode approximation is written as a modified Mathieu equation having damping and non-linear terms which are assumed to be of order ε (non-dimensional excitation amplitude being considered as small). In this equation, the assumed expansion of the response is put and the coefficients of like powers of ε are equated to zeros leading to a set of second order ordinary differential equations. For a bounded solution of these equations, one has to avoid secular terms by making the coefficient of secular terms to be zero. This gives a differential equation (DE) in amplitude 'A' from which the response of the system can be obtained using polar form for 'A'. The real and imaginary parts of this resulting equation are separated into two non-autonomous D.E's which

are converted into autonomous system by appropriately eliminating the time 't'. From the resulting equations the steady state response is found out.

6.1 ANALYSIS:

Equation (2.21) involving independent variable z can be rewritten as

$$\ddot{\bar{B}}(t) + \frac{\epsilon C}{\mu} \dot{\bar{B}}(t)+[\omega_{01}^2+\epsilon (K_{f2}-M_{22}\omega^2) \cos \omega t] \bar{B}(t)$$

$$= \epsilon f(\bar{B}, \dot{\bar{B}}, \ddot{\bar{B}}) \qquad \ldots.(6.1)$$

where $t= z/\omega$, $C = \frac{c}{\epsilon}$,

$$f(\bar{B},\dot{\bar{B}}, \ddot{\bar{B}}) = -\frac{1}{\epsilon} [\delta\bar{B}^3 - \frac{3\pi^2 C}{8\mu} \epsilon \bar{B}^2 \dot{\bar{B}} + M_{46}\bar{B} \dot{\bar{B}}^2$$

$$+ M_{44}\bar{B}^2 \ddot{\bar{B}}] + K_{f2} \bar{B} \qquad \ldots.(6.1a)$$

and overdots denote differentiation with respect to t.

In the method of multiple scales, one introduces the new independent variables T_o and T_1 according to

$$T_n = \epsilon^n t \quad \text{for} \quad n= 0,1 \qquad \ldots.(6.2)$$

$$\therefore \quad \frac{dT_n}{dt} = \epsilon^n$$

So ti follows that the derivatives with respect to time t become expansions in terms of the partial derivatives with respect to T_n according to

$$\frac{d}{dt} = \frac{dT_o}{dt}\frac{\partial}{\partial T_o} + \frac{dT_1}{dt}\cdot\frac{\partial}{\partial T_1}$$

$$= D_o + \epsilon D_1 \qquad \text{.....(6.3)}$$

$$\frac{d^2}{dt^2} = D_o^2 + 2\epsilon D_o D_1 + \epsilon^2 D_1^2$$

The solution of (6.1) can be represented by an expansion of the form

$$\overline{B}(t,\epsilon) = \overline{B}_o (T_o, T_1) + \epsilon \overline{B}_1(T_o, T_1) \qquad \text{.....(6.4)}$$

Where, $T_o = t$ and $T_1 = \epsilon t$.

Substituting (6.3) and (6.4) into (6.1) and equating the coefficients of ϵ^o and ϵ on both sides, the follwoing equations are obtained

$$D_o^2 \overline{B}_o + \omega_{o1}^2 \overline{B}_o = 0 \qquad \text{.....(6.5)}$$

$$D_o^2 \overline{B}_1 + \omega_{o1}^2 \overline{B}_1 = -2D_oD_1 \overline{B}_o - (K_{f2} - M_{22}\omega^2) \operatorname{Cos} \omega t \overline{B}_o$$

$$- \frac{C}{\mu} D_o \overline{B}_o + f(\overline{B}_o, D_o\overline{B}_o, D_o^2 \overline{B}_o) \qquad \text{..... (6.6)}$$

Equation (6.5) is homogeneous linear DE with constant coefficient and (6.6) is non-homogeneous DE.

The solution of (6.5) is written as

$$\overline{B}_o = A(T_1) \exp(i\omega_{o1}T_o) + \overline{A}(T_1) \exp(-i\omega_{o1}T_o)$$

$$\text{..... (6.7)}$$

where, 'A' is an unknown complex function of T_1 and $\bar{A}$ is its complex conjugate. It is seen that $\bar{B}_o$ is real as second term is complex conjugate of the first. Substituting the solution of $\bar{B}_o$ into equation (6.6).

$$D_o^2 \bar{B}_1 + \omega_{o1}^2 \bar{B}_1 = -2i\omega_{o1} A'(T_1) \exp(i\omega_{o1} T_o)$$

$$- (K_{f2} - M_{22}\omega^2) \left[\frac{\bar{A}(T_1)}{2} \cdot (\exp(-i\omega_{o1} T_o + i\omega T_o) \right.$$

$$\left. + \exp(-i\omega_{o1} T_o - i\omega T_o)) \right]$$

$$- \frac{C}{\mu} i\omega_{o1} A(T_1) \exp(i\omega_{o1} T_o) + CC$$

$$+ f[A(T_1) \exp(i\omega_{o1} T_o) + \bar{A}(T_1) \exp(-i\omega_{o1} T_o),$$

$$i\omega_{o1}[A(T_1) \exp(i\omega_{o1} T_o) - \bar{A}(T_1) \exp(-i\omega_{o1} T_o)],$$

$$- \omega_{o1}^2 (A(T_1) \exp(i\omega_{o1} T_o) + \bar{A}(T_1) \exp(-i\omega_{o1} T_o))] \quad \ldots.(6.8)$$

where CC denotes the complex conjugate of the preceding terms and primes denote differentiation with respect to T_1.

To investigate the steady state response in the neighbourhood of principal parameteric resonance write

$$\omega = 2\omega_{o1} + \varepsilon\sigma \quad \ldots.(6.9)$$

where σ = detuning parameter.

$$\therefore (\omega - \omega_{o1}) T_o = \omega_{o1} T_o + \sigma T_1 \quad \ldots. (6.10)$$

Now from the equation (6.8) the solution of $\bar{B}_1$ can be looked for. The particular solution of this equation has

some secular terms, containing the factor $T_o \exp(\pm i\omega_{o1} T_o)$ and in order to get a bounded solution of $\overline{B}_1$ and, therefore, a uniformly valid expansion of the response, these secular terms are to be suppressed. Making the coefficient of this secular term as zero

$$2i\omega_{o1} \; A' = - (K_{f2} - M_{22}\omega^2) \; \frac{\overline{A}}{2} \; \exp(i\sigma T_1)$$

$$- \; \frac{C}{\mu} \, i\omega_{o1} \; A + \frac{\omega_{o1}}{2\pi} \int_0^{2\pi/\omega_{o1}} f \exp(-i\omega_{o1} T_o) dT_o \qquad \text{.... (6.11)}$$

where $f(\overline{B}_o, D_o\overline{B}_o, D_o^2 \overline{B}_o)$ in equation (6.8) is expanded in a Fourier series [7] according to

$$f(\overline{B}_o, D_o\overline{B}_o, D_o^2 \overline{B}_o) = \sum_{n=0}^{\infty} [f_n(\overline{B}_o, D_o\overline{B}_o, D_o^2\overline{B}_o) \exp(in\omega_{o1} T_o) + CC]$$

and thus the coefficient of $\exp(i\omega_{o1} T_o)$ in it is

$$f_1(\overline{B}_o, D_o\overline{B}_o, D_o^2\overline{B}_o) = \frac{\omega_{o1}}{2\pi} \int_0^{2\pi/\omega_{o1}} f(\overline{B}_o, D_o\overline{B}_o, D_o^2 \overline{B}_o) \exp(-i\omega_{o1} T_o) dT_o$$

As 'A' is a complex quantity it can be expressed by

$$A = \frac{1}{2} \, a \exp(i\alpha) \qquad \text{.... (6.12)}$$

where a and α are real.

Putting the equation (6.12) into (6.11) and separating the real and imaginary parts, the following two equations in 'a' and 'α' are obtained

$$a' = - \frac{a}{4\omega_{o1}} (K_{f2} - M_{22}\omega^2) \; \text{Sin}\,(\sigma T_1 - 2\alpha)$$

$$- \; \frac{C}{2\mu} \, a - \frac{1}{2\pi} \int_0^{2\pi/\omega_{o1}} f . \text{Sin}(\omega_{o1} T_o + \alpha) \; dT_o$$

and

$$a\alpha' = \frac{a}{4\omega_{o1}}(K_{f2} - M_{22}\omega^2)\mathrm{Cos}(\sigma T_1 - 2\alpha)$$

$$- \frac{1}{2\pi}\int_0^{2\pi/\omega_{o1}} f.\mathrm{Cos}\ (\omega_{o1}T_o + \alpha)dT_o \qquad \ldots..(6.13)$$

Putting (6.12) into (6.7) , one gets

$$\bar{B}_o = a\ \mathrm{Cos}\ (\omega_{o1}T_o + \alpha)$$

$$= a\ \mathrm{Cos}\ \phi \qquad \ldots.\ (6.14)$$

Where $\phi = \omega_{o1}T_o + \alpha$

Therefore, the zeroth order approximation for $\bar{B}$ is given by

$$\bar{B} = a\ \mathrm{Cos}\ \phi + O\ (\epsilon) \qquad \ldots.\ (6.14a)$$

The equations (6.13) are non-autonomous as time T_1 appears explicitly in these equations. Substituting

$$\psi = \sigma T_1 - 2\alpha$$

in equations (6.13) and thus converting these into an autonomous system of D.E.s

$$a' = -\frac{a}{4\omega_{o1}}(K_{f2} - M_{22}\omega^2)\ \mathrm{Sin}\psi - \frac{Ca}{2\mu}$$

$$- \frac{1}{2\pi\omega_{o1}}\int_0^{2\pi} f.\ \mathrm{Sin}\ \phi\ \ d\phi$$

and

$$\frac{a\psi'}{2} = \frac{a\sigma}{2} - \frac{a}{4\omega_{o1}}(K_{f2}-M_{22}\omega^2)\text{Cos }\psi$$

$$+ \frac{1}{2\pi\omega_{o1}} \int_o^{2\pi} f.\ \text{Cos }\phi\ d\phi \qquad \ldots(6.15)$$

where $\frac{\psi'}{2} = \frac{\sigma}{2} - \alpha'$.

For its steady state solution

$$a' = \psi' = 0 \qquad \ldots(6.16)$$

Put the zeroth order approximation of $\overline{B}$ (given by equation (6.14a) into (6.1a), integrate the definite integrals in equations (6.15) and use the condition (6.16) to get

$$\frac{1}{4\omega_{o1}}(K_{f2} - M_{22}\omega^2)\text{Sin }\psi = \frac{-c}{2\mu E}$$

$$\ldots.\ (6.17)$$

and

$$\frac{1}{4\omega_{o1}}(K_{f2}-M_{22}\omega^2)\ \text{Cos }\psi = \frac{\sigma}{2} + \frac{K_{f2}}{2\omega_{o1}} - \frac{a^2}{2E\omega_{o1}}\left[\frac{3\delta}{4} + \frac{M_{46}-3M_{44}}{4}.\omega_{o1}^2\right]$$

Eliminating ψ from equations (6.17), one gets an equation for amplitude 'a' of the response as a function of the detuning parameter σ and the amplitude of the excitation E, called the frequncy response equation. On multiplying the resulting equation by (E^2/ω_{o1}^2) on both sides

$$\left(\frac{\epsilon K_{f2}}{4\omega_{o1}^2} - \epsilon M_{22}q^2\right)^2 = \left(\frac{c}{2\mu\omega_{o1}}\right)^2$$

$$+\left[\frac{\epsilon K_{f2}}{2\omega_{o1}^2} + q-1 - \frac{a^2}{2}(\delta_* - K)\right]^2 \qquad \text{.....(6.18)}$$

where,

$$q = \frac{\omega}{2\omega_{o1}} = 1 + \frac{\epsilon\,\sigma}{2\omega_{o1}},$$

$$\delta_* = \frac{3\delta}{4\omega_{o1}^2}$$

and $$K = \frac{3M_{44}-M_{46}}{4}.$$

For zero damping (c=0)

$$a \doteq \left[\frac{\mu_* + q - 1 \mp \left(\frac{\mu_*}{2} - \epsilon\, M_{22}q^2\right)^{1/2}}{\frac{1}{2}(\delta_* - K)}\right]^{1/2} \qquad \text{.....(6.19)}$$

The backbone curves are obtained from equation (6.1) by replacing $\frac{\omega}{2}$ by ω , following the above procedure with $\omega=\omega_{o1}+\epsilon\sigma$ and equating the terms to zero which correspond to damping and displacement excitation. This is equivalent to setting $c=\epsilon=0$ in equation (6.18), where q is to be interpreted as $\frac{\omega}{\omega_{o1}}$:

$$a = \left[\frac{q-1}{\frac{1}{2}(\delta_* - K)}\right]^{1/2} \qquad \text{.....(6.20)}$$

The expressions for δ_* and K are same as those in the equation (6.18).

6.2 RESULTS:

The backbone curves are plotted for the following two cases from the equation (6.20)

(i) $\varepsilon = 0$, $M_* = 0.1$, $\frac{F_o}{P_{cr}} = 0.5$, $\delta_d = 0$ and K_{f*} as a parameter ranging from 0.0 to 50.0 (See Fig. 6a)

(ii) $\varepsilon = 0$, $K_{f*} = 15$, $\frac{F_o}{P_{cr}} = 0.5$, $\delta_d = 0$ and M_* as a parameter ranging from 0.0 to 0.4 (See Fig. 6b).

The response curves are plotted from equation (6.19) for the following cases taking only (-) sign in place of ($\pm$) sign in it

(i) $\varepsilon = 0.05$, $M_* = 0.1$, $\delta_d = 0$, $\frac{F_o}{P_{cr}} = 0.5$ and stiffness ratio K_{f*} as a parameter ranging from 0.0 to 50.0 (See Fig.6c)

(ii) $\varepsilon = 0.05$, $K_{f*} = 2$, $\delta_d = 0$, $\frac{F_o}{P_{cr}} = 0.5$ and mass ratio M_* as a parameter ranging from 0.0 to 0.4 (See Fig. 6d)

(iii) $M_* = 0.1$, $K_{f*} = 15$, $\hat{\delta}_d = 0$, $\frac{F_o}{P_{cr}} = 0.5$ and excitation amplitude ε as a parameter ranging from 0.00 to 0.05 (See Fig. 6e).

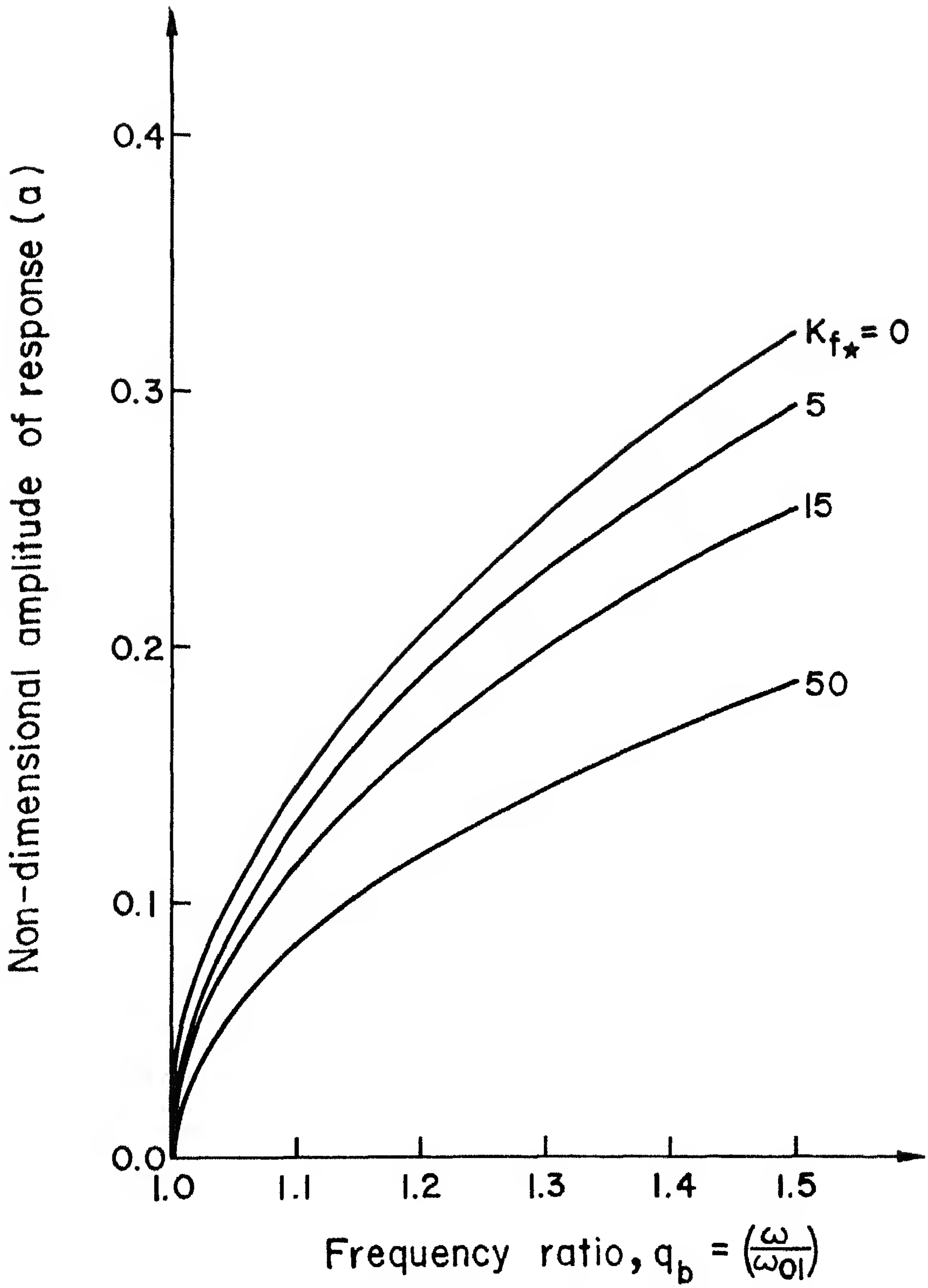

Fig. 6a Backbone curve for various stiffness ratio with mass ratio $M_\star = 0.1$, $\epsilon = 0$ $\frac{F_0}{P_{cr}} = 0.5$, $\delta_d = 0$

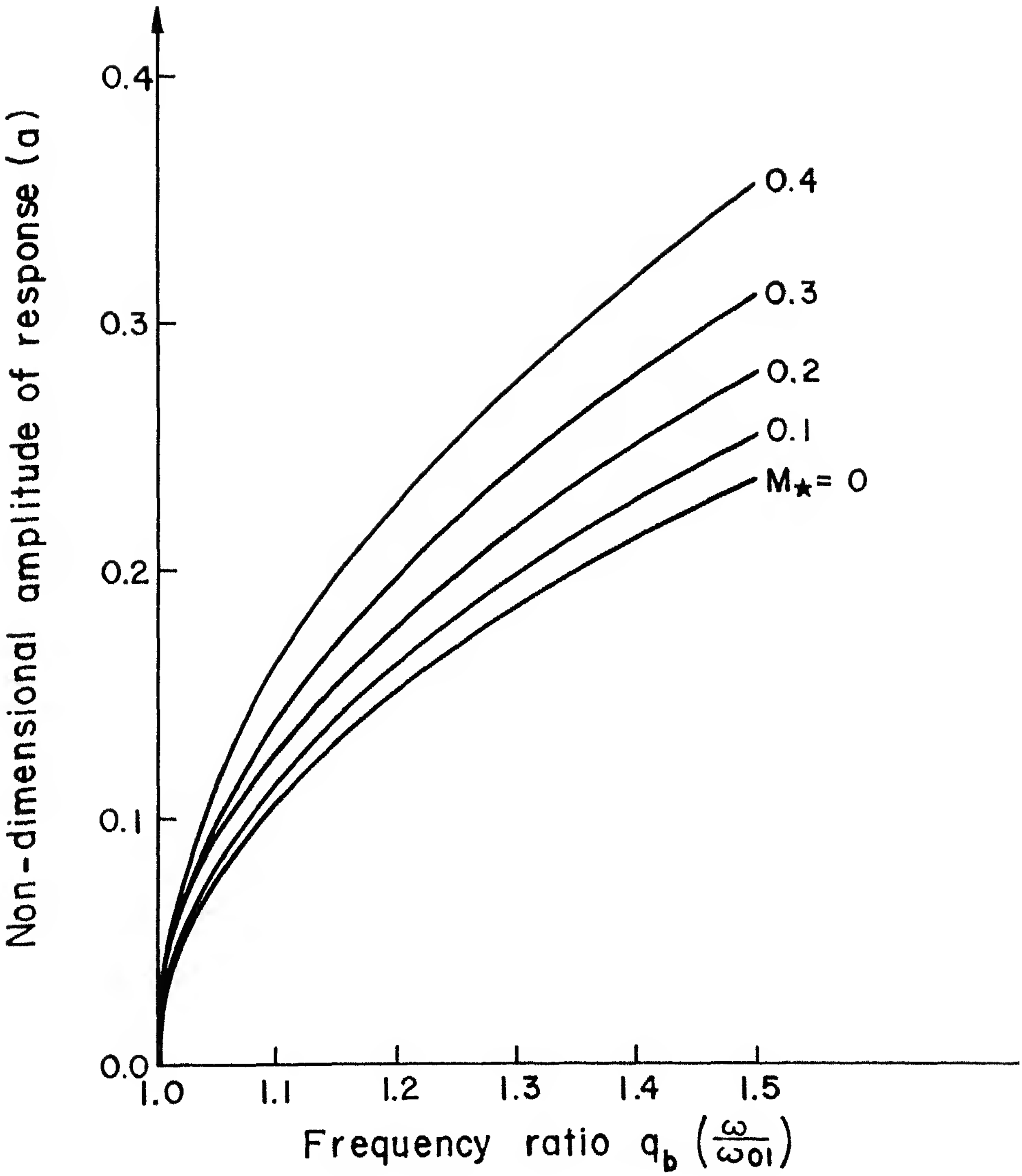

Fig. 6b Backbone curve for various mass ratios with $K_{f\star} = 15$, $\epsilon = 0$, $\frac{F_0}{P_{cr}} = 0.5$, $\delta_d = 0$

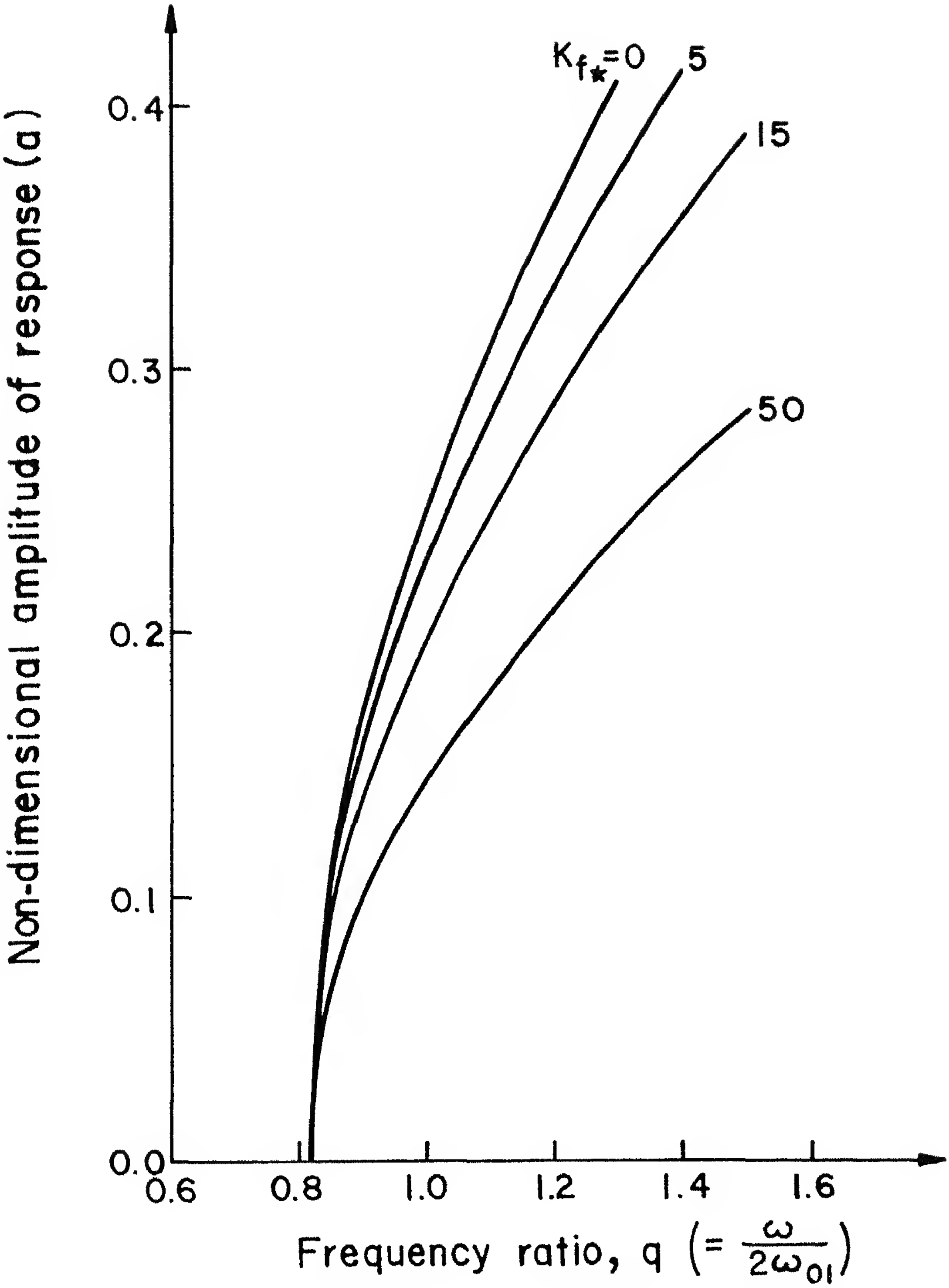

Fig. 6C Response curves for various stiffness ratio
with $\epsilon = 0.05$, $M_* = 0.1$, $\frac{F_0}{P_{cr}} = 0.5$, $\delta_d = 0$

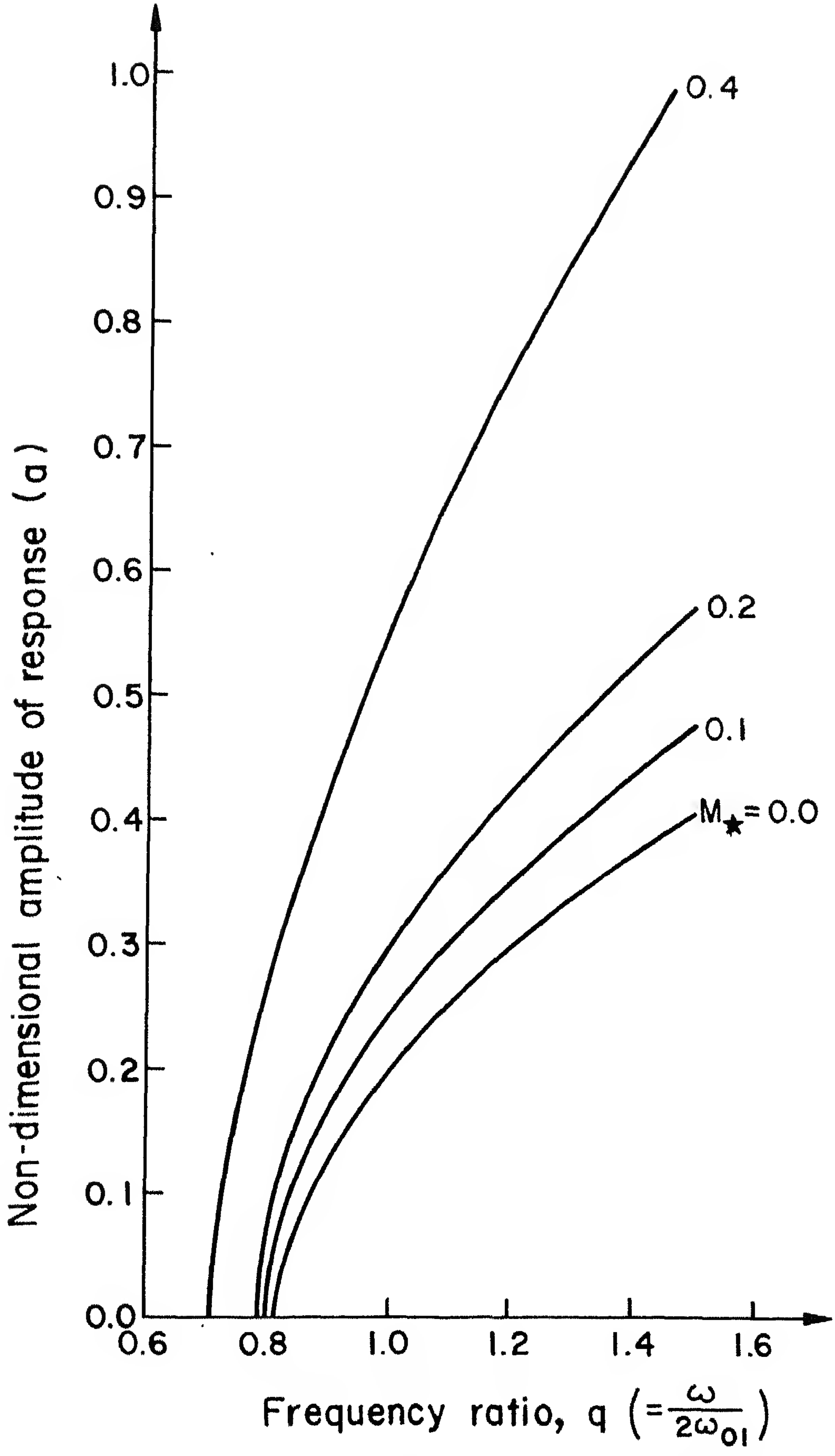

Fig. 6d. Response curves for various mass ratios with $\epsilon = 0.05$, $K_{f\star} = 2$, $\frac{F_0}{P_{cr}} = 0.5$, $\delta_d = 0$

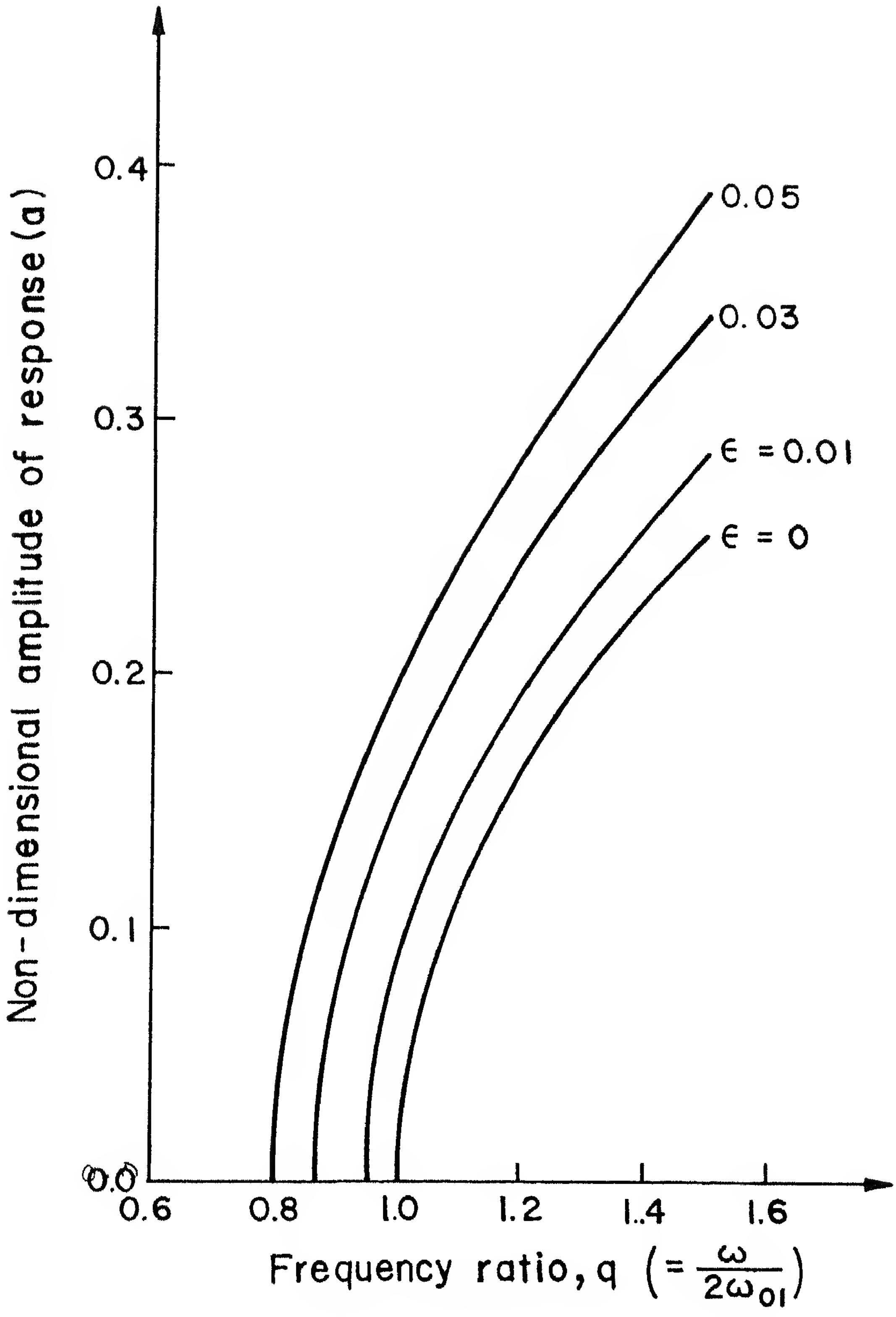

Fig. 6e Response curves for various excitation amplitudes with $M_{\star} = 0.1$, $K_{f\star} = 15$, $\frac{F_0}{P_{cr}} = 0.5$, $\delta_d = 0$

CHAPTER 7

DISCUSSIONS

The principal parametric instability region for the idealised engine-valve mechanism for mass ratio $M_* = 0.1$, stiffness ratio $K_{f*} = 50$, ratio of the preload in the spring to the beam critical load $\frac{F_o}{P_{cr}} = 0.0$ and for one mode approximation with no damping (δ_d) is shown in Fig.3a. For small values of Θ, the transition curves are linear and comparable with that by Gurgoze[14] but for higher values of Θ there is a side tilt from the linear transition curve. For the same system parameters and one mode approximation with damping $\delta_d = 0.1$, it is observed that this region is lifted up as was noticed by Gurgoze[14]. Considering three terms in the Fourier series representation of the response and taking the same system parameters, this region is relocated for $\delta_d = 0.0$ and 0.1. Comparing the transition curves for one term and three terms of the Fourier series expansion for B(t), it is observed that the right side droops down with marginal change in the left side. Taking two mode approximation the transition curves starting from q = 0.5, 1.0, 2.0 and other admissible discrete values equal to and less than 4 are plotted in Fig.3(b) with the same system parameters and no damping. It is observed that the principal paramatric region

is reduced and the other instability regions are narrow .

The backbone curves found by the method of slowly varying parameters (SVP) are plotted in Fig.5(a) for constant values of the system parameters $M_* = 0.1$, $\varepsilon = 0$, $\frac{F_o}{P_{cr}} = 0.5$, $\delta_d = 0$ and stiffness ratio K_{f*} varying from 0.0 to 50.0. Similar curves are found for the system parameters $K_{f*} = 0$, $\varepsilon = 0$, $\frac{F_o}{P_{cr}} = 0.5$, $\delta_d = 0$ and various values of M_* (0.0 - 0.4). It is observed that the nature of the change in these curves for different system parameters by this method is similar to that found by application of harmonic balance (H.B) method by Gurgoze and also by the application of method of multiple scale (M.S.). The response curves (stable and unstable branches) found out by method of S.V.P. for the system parameters $\varepsilon = 0.05$, $\frac{F_o}{P_{cr}} = 0.5$, $\delta_d = 0$, $M_* = 0.1$ and various values of K_{f*} (0.0-50.0) are shown in Fig.5(c). Similarly response curves are shown in Figs.5(b) and 5(e) to show the effect of system parameters M_* and ε . It is observed that the nature of these curves is similar to that found by the method of M.S. in Chapter -6 and also to that found by the method of H.B. done by Gurgoze. Backbone curves and response curves found by these three methods are compared in Figs.7(a -e).

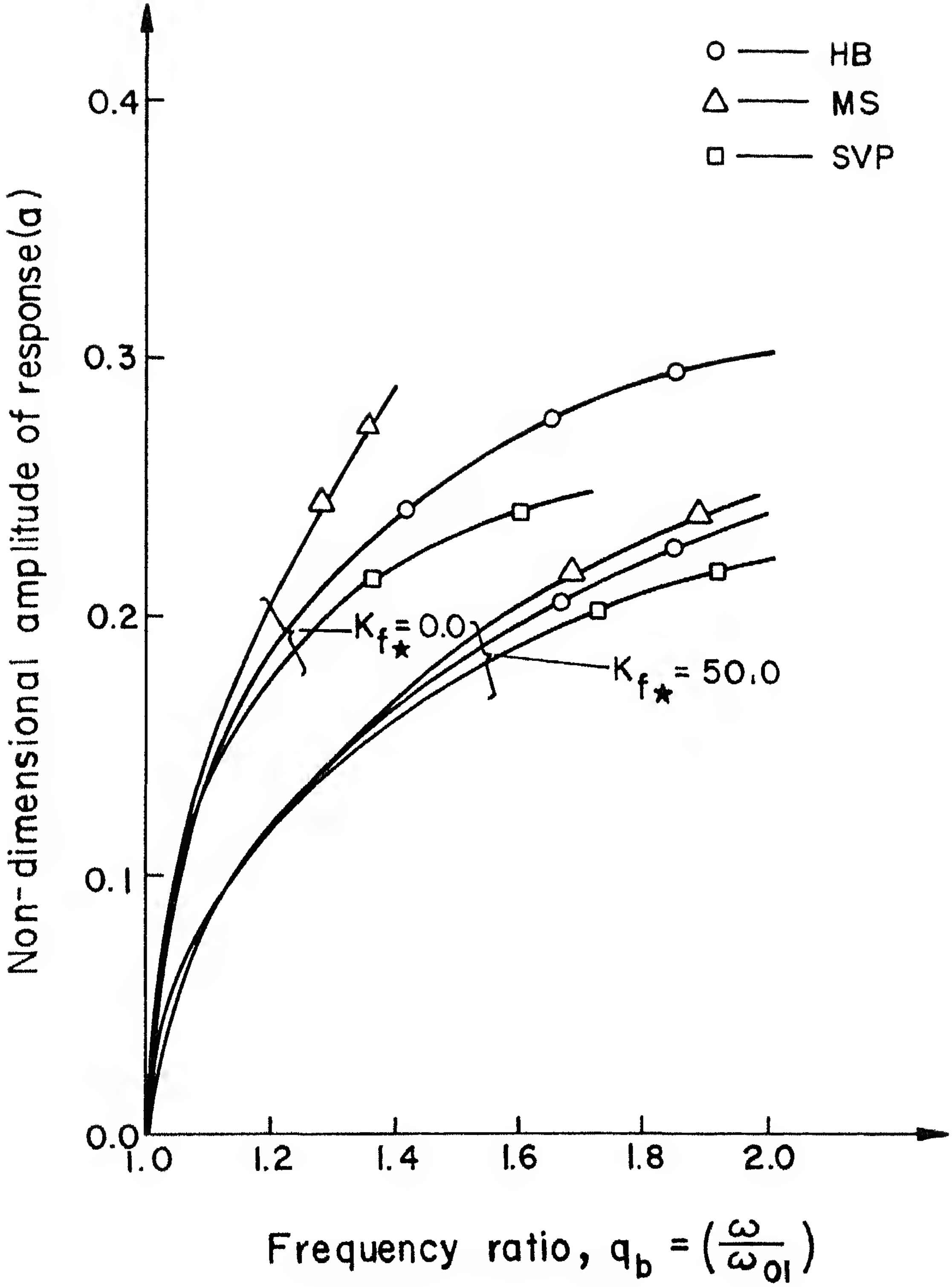

Fig. 7.a Comparison of Backbone curves obtained by three different methods with $M_{\star} = 0.1$, $\epsilon = 0.0$, $\delta_d = 0.0$ and $\frac{F_0}{P_{cr}} = 0.5$

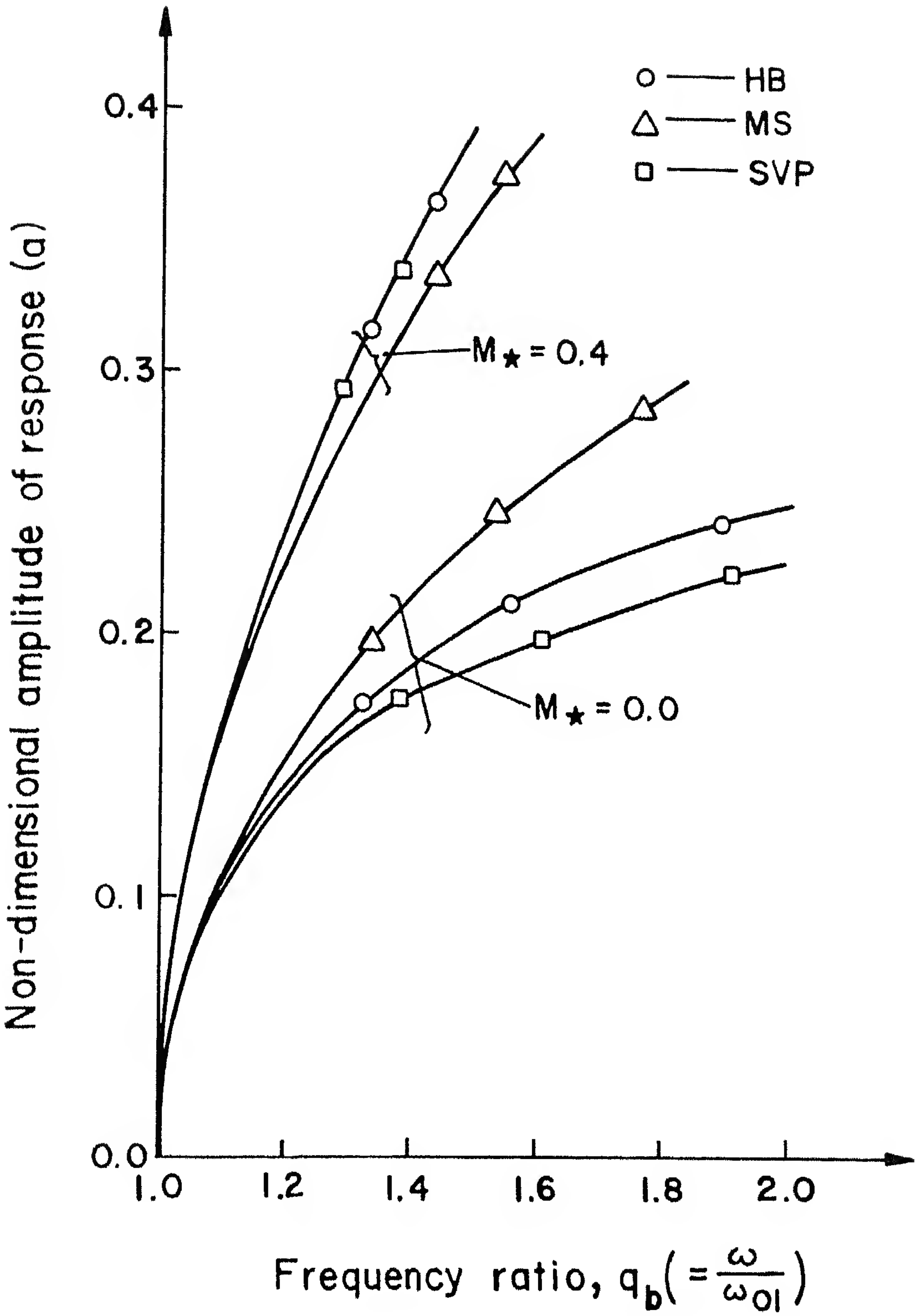

Fig. 7b Comparison of Backbone curves obtained by three different methods for $K_{f\star} = 15.0$, $\epsilon = 0.0$, $\delta_d = 0.0$ and $\frac{F_0}{P_{cr}} = 0.5$

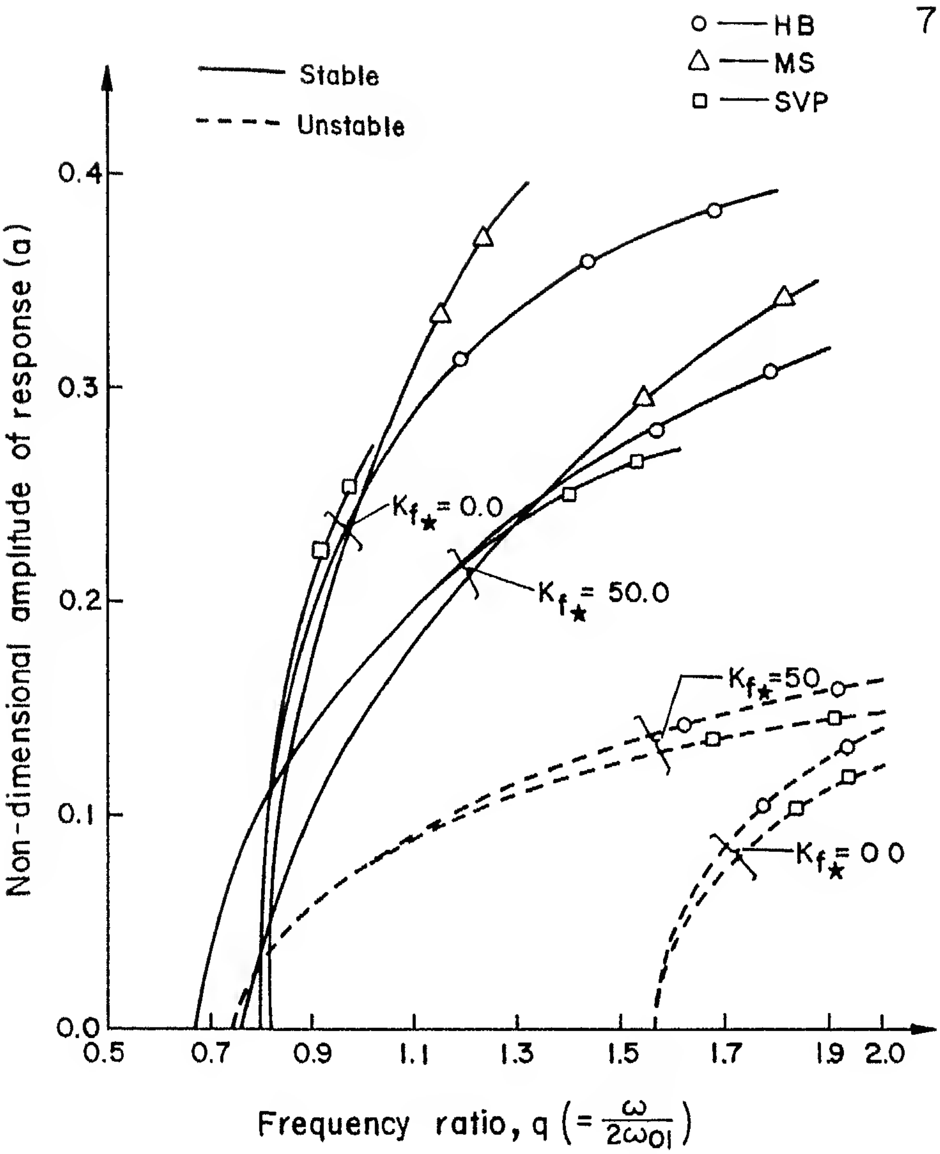

Fig. 7c Comparison of response curves by three different methods for $M_\star = 0.1$, $\epsilon = 0.05$, $\delta_d = 0.0$ and $\frac{F_0}{P_{cr}} = 0.5$

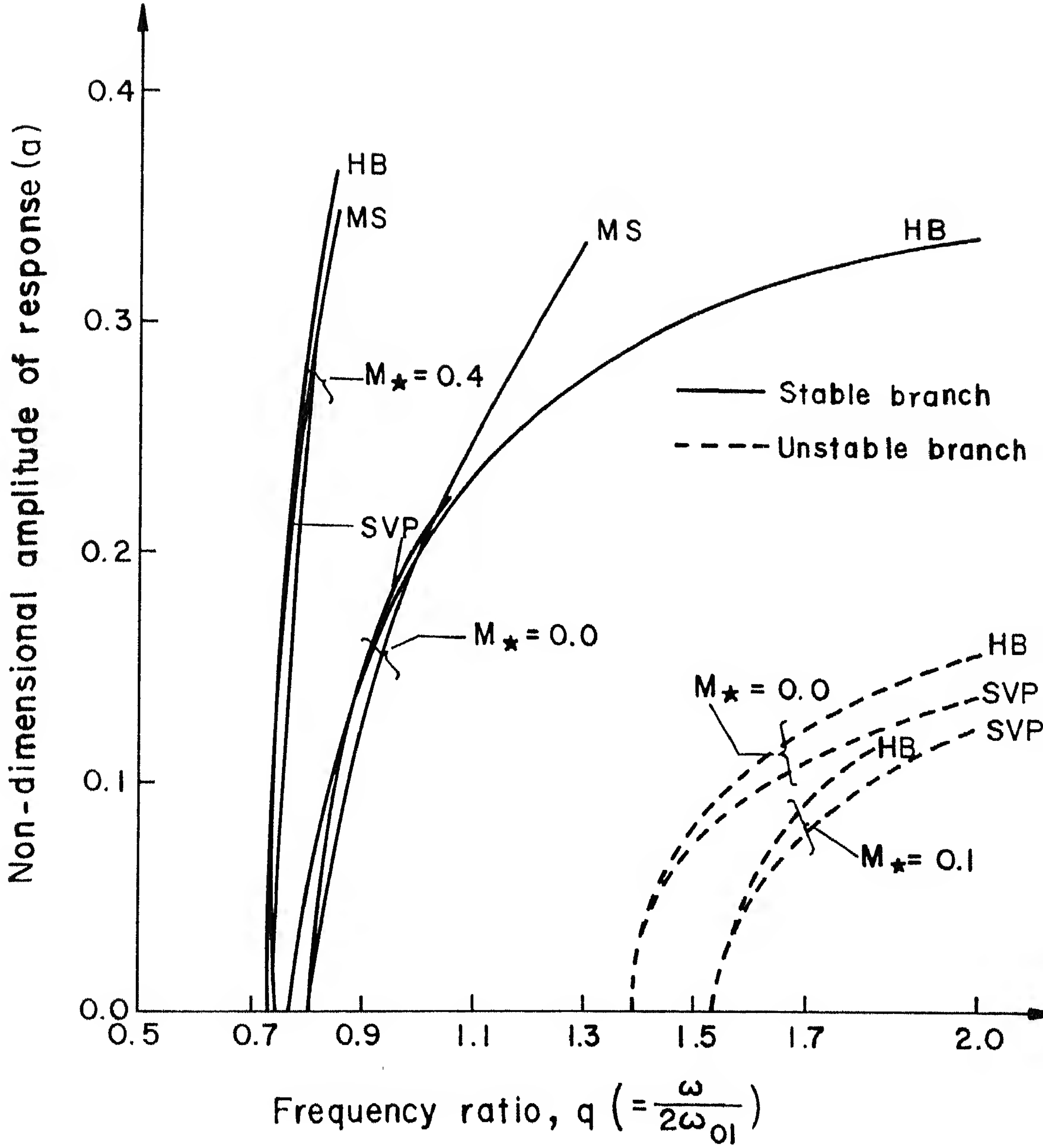

Fig. 7d Comparison of response curves by three different methods for $\epsilon = 0.05$, $K_{f\star} = 2.0$ $\delta_d = 0.0$ and $\frac{F_o}{P_{cr}} = 0.5$

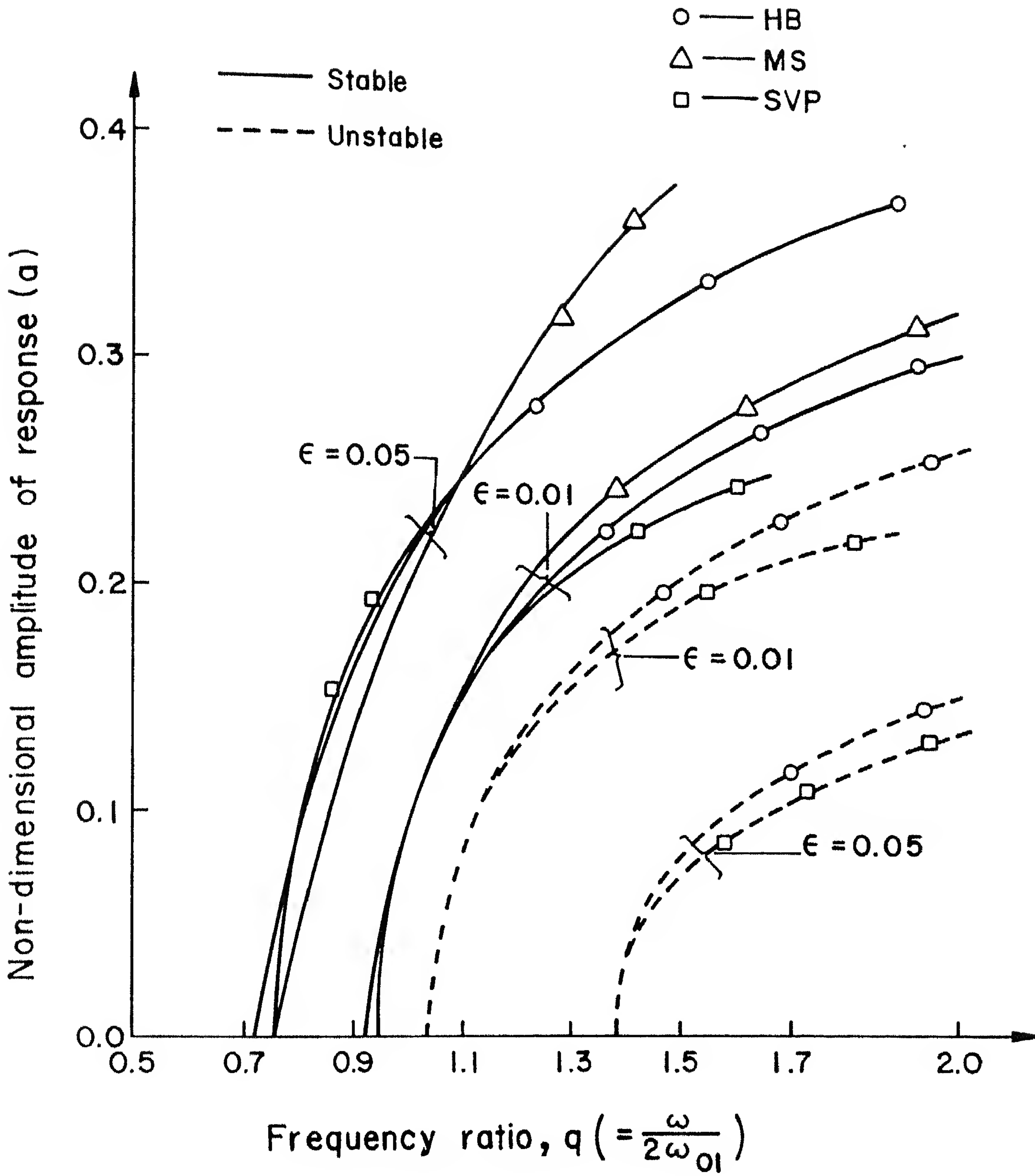

Fig. 7e Comparison of response curves by three different methods for $M_\star = 0.1$, $K_{f\star} = 15.0$ $\delta_d = 0.0$ and $\frac{F_0}{P_{cr}} = 0.5$

CHAPTER 8

CONCLUSION

The non-linear parametric vibrations of a restrained vertical beam with a concentrated end mass at the top under harmonic ground excitation has been studied. The instability region(s) of the linearised system for one mode and two mode approximation is/ are found out. The steady state response curves are obtained by the method of

(i) harmonic balance,

(ii) slowly varying parameters

and (iii) multiple scales.

The major findings are summarized below.

The principal instability boundaries bifurcating from $\frac{\omega}{2\omega_{01}} = 1$ are not linear for higher values of the excitation amplitude. The effect of the second mode on these boundaries is observed to reduce the instability region.

For a fixed excitation frequency

(i) an increase in the stiffness ratio decreases the amplitude of the steady state response

(ii) an increase in the mass ratio increases the steady state amplitude and

(iii) an increase in the amplitude of the ground excitation increases the steady state amplitude.

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APPENDIX -I

Combining equations (2.1) and (2.2)

$$- F(t)\ v - M_1 + M_2 - M_3 = EIv_{,ss}(1-v^2_{,s})^{-1/2} \qquad \ldots\ldots \quad (1.1)$$

Expand the right hand side of the above equation and keep only upto third order principal non-linear terms to get

$$- F(t)\ v - M_1 + M_2 - M_3 = EI\ v_{,ss}(1+ \tfrac{1}{2} v^2_{,s}). \qquad \ldots\ldots \quad (1.2)$$

Differentiating the above equation twice with respect to s

$$- F(t)\ v_{,ss} - \frac{\partial^2 M_1}{\partial s^2} - \frac{\partial^2 M_2}{\partial s^2} - \frac{\partial^2 M_3}{\partial s^2}$$

$$= EI\ [v_{,ssss} + \tfrac{1}{2} v_{,ssss} v^2_{,s} + 3v_{,s} v_{,ss} v_{,sss} + v^3_{,ss}]. \qquad \ldots\ldots \quad (1.3)$$

From the equation (2.9)

$$M_1 = \int_s^L \mu v_{,tt} [\int_s^\beta (1- \tfrac{1}{2} v^2_{,\eta})\, d\eta]\, d\beta$$

$$- R \int_s^L (1- \tfrac{1}{2} v^2_{,\beta})\, d\beta \qquad \ldots\ldots \quad (1.4)$$

In equation (2.7) M_{11} is defined as

$$M_{11} = \int_s^L \mu v_{,tt} [\int_s^\beta (1-\tfrac{1}{2} v^2_{,\eta})d\eta]\, d\beta \qquad \ldots\ldots \quad (1.5)$$

$$= \int_s^L f(\beta, s) d\beta,$$

where $f(\beta, s) = \mu v_{,tt} \int_s^\beta (1- \frac{1}{2} v^2_{,\eta})d\eta$

Differentiating M_{11} partially with respect to s by Leibnitz rule

$$\frac{\partial M_{11}}{\partial s} = \frac{\partial}{\partial s} \int_s^L f(\beta,s)d\beta$$

$$= \int_s^L \frac{\partial}{\partial s} f(\beta,s)\, d\beta - f(s,s)$$

$$= \int_s^L \frac{\partial}{\partial s} \left[\mu v_{,tt} \int_s^\beta (1- \tfrac{1}{2} v_{;\eta}^2)\, d\eta\right] d\beta$$

$$- \mu v_{,tt} \int_s^s (1- \tfrac{1}{2} v_{,\eta}^{\,2})\, d\eta$$

$$= \int_s^L \mu v_{,tt} \left[\int_s^\beta \frac{\partial}{\partial s} (1- \tfrac{1}{2} v_{,\eta}^{\,2})\, d\eta\right.$$

$$\left. - (1- \tfrac{1}{2} v_{,s}^2) + (1-\tfrac{1}{2} v_{,\beta}^2) \frac{d\beta}{ds} \right] d\beta$$

$$=- \int_s^L \mu v_{,tt}(1- \tfrac{1}{2} v_{,s}^2)\, d\beta$$

$$=- (1-\tfrac{1}{2} v_{,s}^{\,2}) \int_s^L \mu v_{,tt}\, d\beta \qquad \ldots\ldots (1.6)$$

Again differentiating $\dfrac{\partial M_{11}}{\partial s}$ partially with respect to s

$$\frac{\partial^2 M_{11}}{\partial s^2} = - (1- \tfrac{1}{2} v_{,s}^2) \frac{\partial}{\partial s} \int_s^L \mu\, v_{,tt}\, d\beta$$

$$+ v_{,s}\, v_{,ss} \int_s^L \mu v_{,tt}\, d\beta$$

$$=\mu v_{,tt}(1- \tfrac{1}{2} v_{,s}^{2}) + v_{,s}\, v_{,ss} \int_s^L \mu v_{,tt}\, d\beta \qquad \ldots\ldots (1.7)$$

Therefore, one gets

$$\frac{\partial^2 M_1}{\partial s^2} = \frac{\partial^2 M_{11}}{\partial s^2} - \frac{\partial^2}{\partial s^2}\left[R \int_s^L (1 - \tfrac{1}{2} v_{,\beta}^2)\, d\beta\right.$$

$$= \mu v_{,tt}\,(1 - \tfrac{1}{2} v_{,s}^2) + v_{,s} v_{,ss} \int_s^L \mu v_{,tt}\, d\beta - R v_{,s}\, v_{,ss} \qquad \ldots\ldots (1.8)$$

[As 'R' is constant]

From the equation (2.10)

$$M_2 = \int_s^L \mu \left[g_{,tt} + \int_0^\beta (v_{,\eta t}^2 + v_{,\eta}\, v_{,\eta tt})\, d\eta\right].\left(\int_s^\beta v_{,\eta} d\eta\right) d\beta \qquad \ldots\ldots (1.9)$$

Differentiating M_2 partially with respect to s

$$\frac{\partial M_2}{\partial s} = \int_s^L \frac{\partial}{\partial s}\left[\mu\left(g_{,tt} + \int_0^\beta (v_{,\eta t}^2 + v_{,\eta}\, v_{,\eta tt})\, d\eta\right)\right.$$

$$\left.\left(\int_s^\beta v_{,\eta}\, d\eta\right)\right] d\beta$$

$$-\mu\left[g_{,tt} + \int_0^s (v_{,\eta t}^2 + v_{,\eta}\, v_{,\eta tt})\, d\eta\right]\left(\int_s^s v_{,\eta} d\eta\right)$$

$$= - v_{,s} \int_s^L \left[\mu\left(g_{,tt} + \int_0^\beta (v_{,\eta t}^{\,2} + v_{,\eta}\, v_{,\eta tt})\, d\eta\right]\right. d\beta \qquad \ldots\ldots (1.10)$$

Again differentiating $\frac{\partial M_2}{\partial s}$ partially with respect to s

$$\frac{\partial^2 M_2}{\partial s^2} = \mu v_{,s}\left[g_{,tt} + \int_0^s (v_{,nt}^2 + v_{,\eta}\, v_{,ntt}) d\eta\right]$$

$$-\mu v_{,ss} \int_s^L \left[g_{,tt} + \int_0^\beta (v_{;\eta t}^2 + v_{,\eta} v_{,\eta tt})\, d\eta\right] d\beta \qquad \ldots\ldots (1.11)$$

From the equation (2.11)

$$M_3 = \int_s^L cv_{,t} \left[\int_s^\beta (1- \tfrac{1}{2} v_{,\eta}^2)d\eta\right] d\beta \qquad \ldots.(1.12)$$

Observe that M_3 can be obtained by substituting c for μ and $v_{,t}$ for $v_{,tt}$ in the expression for M_{11} (given by equation (1.5)). Therefore, substitute c for μ, $v_{,t}$ for $v_{,tt}$ and M_3 for M_{11} in equation (1.7) to get

$$\frac{\partial^2 M_3}{\partial s^2} = cv_{,t} (1- \tfrac{1}{2} v_{,s}^2) + v_{,s}v_{,ss} \int_s^L cv_{,t}\, d\beta \qquad \ldots.. (1.13)$$

Substitute equations (1.8),(1.11) and (1.13) in the equation (1.3) and simplify to get the equation (2.14).